The Green-Tao theorem
and
a relative Szemerédi theorem

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Green–Tao Theorem (arXiv 2004; Annals 2008)

The primes contain arbitrarily long arithmetic progressions.

Examples:

- 3, 5, 7
- 5, 11, 17, 23, 29
- 7, 37, 67, 97, 127, 157
- Longest known: 26 terms
Green–Tao Theorem (2008)
The primes contain arbitrarily long arithmetic progressions (AP).

Szemerédi’s Theorem (1975)
Every subset of \( \mathbb{N} \) with positive density contains arbitrarily long APs.

(upper) density of \( A \subset \mathbb{N} \) is \( \limsup_{N \to \infty} \frac{|A \cap [N]|}{N} \)

\([N] := \{1, 2, \ldots, N\}\)

\(P = \text{prime numbers}\)

Prime number theorem: \( \frac{|P \cap [N]|}{N} \sim \frac{1}{\log N} \)
Proof strategy of Green–Tao theorem

$P =$ prime numbers, $Q =$ “almost primes”

$P \subseteq Q$ with relative positive density, i.e., $\frac{|P \cap [N]|}{|Q \cap [N]|} > \delta$
Proof strategy of Green–Tao theorem

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Step 1:

Relative Szemerédi theorem (informally)

If \( S \subseteq \mathbb{N} \) satisfies certain pseudorandomness conditions, then every subset of \( S \) of positive density contains long APs.

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Step 1: Relative Szemerédi theorem (informally)

If $S \subset \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of $S$ of positive density contains long APs.

Step 2: Construct a superset of primes that satisfies the conditions.
Relative Szemerédi theorem

Relative Szemerédi theorem (informally)

If $S \subseteq \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of $S$ of positive density contains long APs.

What pseudorandomness conditions?

Green–Tao:

1. Linear forms condition
2. Correlation condition
Relative Szemerédi theorem

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What pseudorandomness conditions?

Green–Tao:
1. Linear forms condition
2. Correlation condition

A natural question (e.g., asked by Green, Gowers, . . . )

Does relative Szemerédi theorem hold with weaker and more natural hypotheses?
Relative Szemerédi theorem

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What pseudorandomness conditions?

Green–Tao:
1. Linear forms condition
2. Correlation condition \( \leftarrow \) no longer needed

A natural question (e.g., asked by Green, Gowers, . . .)
Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

Our main result
Yes! A weak linear forms condition suffices.
Szemerédi’s theorem
   Host set: $\mathbb{N}$

Relative Szemerédi theorem
   Host set: some sparse subset of integers

Conclusion: relatively dense subsets contain long APs
Szemerédi’s theorem

Host set: \( \mathbb{N} \)

Relative Szemerédi theorem

Host set: some sparse subset of integers

Random host set
- Kohayakawa–Łuczak–Rödl ’96
  \[ 3\text{-AP, } p \gtrsim N^{-1/2} \]
- Conlon–Gowers ’10+
- Schacht ’10+

Pseudorandom host set
- Green–Tao ’08 \( \text{linear forms + correlation} \)
- Conlon–Fox–Z. ’13+ \( \text{linear forms} \)

Conclusion: relatively dense subsets contain long APs
Roth’s theorem

**Roth’s theorem (1952)**

If \( A \subseteq [N] \) is 3-AP-free, then \( |A| = o(N) \).

\([N] := \{1, 2, \ldots, N\}\)

3-AP = 3-term arithmetic progression

It’ll be easier (and equivalent) to work in \( \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z} \).
Proof of Roth’s theorem

Roth’s theorem (1952)
If \( A \subseteq \mathbb{Z}_N \) is 3-AP-free, then \( |A| = o(N) \).

Given \( A \), construct tripartite graph \( G_A \) with vertex sets \( X = Y = Z = \mathbb{Z}_N \).
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\( G_A \)
\( x \sim y \) iff
\( 2x + y \in A \)

No triangles? Only triangles \( \iff \) trivial 3-APs with diff 0.

Every edge of the graph is contained in exactly one triangle (the one with \( x + y + z = 0 \)).
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Triangle $xyz$ in $G_A \iff 2x + y, x - z, -y - 2z \in A$

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It’s a 3-AP with diff $-x - y - z$
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If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Constructed a graph with
- $3N$ vertices
- $3N|A|$ edges
- every edge in exactly one triangle

Theorem (Ruzsa & Szemerédi ‘76)
If every edge in a graph $G = (V, E)$ is contained in exactly one triangle, then $|E| = o(|V|^2)$.

(a consequence of the triangle removal lemma)
Proof of Roth’s theorem

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So $3N|A| = o(N^2)$. Thus $|A| = o(N)$. 
Relative Roth theorem

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If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Relative Roth theorem (Conlon, Fox, Z.)
If $S \subseteq \mathbb{Z}_N$ satisfies the 3-linear forms condition, and $A \subseteq S$ is 3-AP-free, then $|A| = o(|S|)$. 

3-linear forms condition:
$G_S$ has asymptotically the expected number of embeddings of $K_{2,2,2}$, $K_{2,2,2}$ & its subgraphs (compared to random graph of same density), e.g.,
Relative Roth theorem

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\[ G_S \]

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Analogy with quasirandom graphs

Chung-Graham-Wilson ’89 showed that in constant edge-density graphs, many quasirandomness conditions are equivalent, one of which is having the correct $C_4$ count

\[
\begin{array}{c}
\text{2-blow-up} \\
\rightarrow
\end{array}
\]

(\text{not shown})
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\[
\begin{align*}
\text{2-blow-up} & \quad \rightarrow \\
\text{\hspace{2cm}} & \quad \rightarrow \\
\end{align*}
\]

In sparse graphs, the CGW equivalences do not hold.
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In sparse graphs, the CGW equivalences do not hold.

Our results can be viewed as saying that:
Many extremal and Ramsey results about $H$ (e.g., $H = K_3$) in sparse graphs hold if there is a host graph that behaves pseudorandomly with respect to counts of the 2-blow-up of $H$. 

\[
\begin{array}{c}
\text{2-blow-up} \\
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\]
Relative Szemerédi theorem (Conlon, Fox, Z.)

Fix $k \geq 3$. If $S \subseteq \mathbb{Z}_N$ satisfies the $k$-linear forms condition, and $A \subseteq S$ is $k$-AP-free, then $|A| = o(|S|)$. 

4-linear forms condition: correct count of the 2-blow-up of the simplex $K(3)^4$ (as well as its subgraphs)
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$k = 4$: build a 4-partite 3-uniform hypergraph

Vertex sets $W = X = Y = Z = \mathbb{Z}_N$

- $xyz \in E \iff 3w + 2x + y \in S$
- $wxz \in E \iff 2w + x - z \in S$
- $wyz \in E \iff w - y - 2z \in S$
- $xyz \in E \iff -x - 2y - 3z \in S$

common diff: $-w - x - y - z$
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common diff: $-w - x - y - z$

4-linear forms condition: correct count of the 2-blow-up of the simplex $K_4^{(3)}$ (as well as its subgraphs)
Two approaches

Conlon, Fox, Z.
A relative Szemerédi theorem
20pp

Z.
An arithmetic transference proof of a relative Szemerédi thm
6pp
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Roth’s theorem: from one 3-AP to many 3-APs

Roth’s theorem

∀δ > 0, for sufficiently large N,
every \( A \subset \mathbb{Z}_N \) with \( |A| \geq \delta N \) contains a 3-AP.
Roth’s theorem: from one 3-AP to many 3-APs

Roth’s theorem
\[ \forall \delta > 0, \text{ for sufficiently large } N, \]
every \( A \subset \mathbb{Z}_N \) with \( |A| \geq \delta N \) contains a 3-AP.

By an averaging argument (Varnavides), we get many 3-APs:

Roth’s theorem (counting version)
\[ \forall \delta > 0 \exists c > 0 \text{ so that for sufficiently large } N, \]
every \( A \subset \mathbb{Z}_N \) with \( |A| \geq \delta N \) contains at least \( cN^2 \) many 3-APs.
Transference

Start with

\[(\text{sparse}) \quad A \subset S \subset \mathbb{Z}_N, \quad |A| \geq \delta |S|\]
Transference

Start with

\[(\text{sparse})\quad A \subset S \subset \mathbb{Z}_N, \quad |A| \geq \delta |S|\]

One can find a dense model $\tilde{A}$ for $A$:

\[(\text{dense})\quad \tilde{A} \subset \mathbb{Z}_N, \quad \frac{|\tilde{A}|}{N} \approx \frac{|A|}{|S|} \geq \delta\]
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Start with

\[ A \subset S \subset \mathbb{Z}_N, \quad |A| \geq \delta |S| \]

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Counting lemma will tell us that

\[ \left( \frac{N}{|S|} \right)^3 |\{3-APs \text{ in } A\}| \approx |\{3-APs \text{ in } \tilde{A}\}| \]
Transference

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\left(\frac{N}{|S|}\right)^3 |\{3-\text{APs in } A\}| \approx |\{3-\text{APs in } \tilde{A}\}| \\
\geq cN^2 \quad \text{[By Roth’s Theorem]}
\]

$\implies$ relative Roth theorem
Roth’s theorem (counting version)

∀δ > 0 ∃c > 0 so that for sufficiently large N, every \( A \subset \mathbb{Z}_N \) with \(|A| \geq \delta N\) contains at least \( cN^2 \) many 3-APs.
Roth’s theorem (counting version)

∀δ > 0 ∃c > 0 so that for sufficiently large N,
every $A \subset \mathbb{Z}_N$ with $|A| \geq \delta N$ contains at least $cN^2$ many 3-APs.

Roth’s theorem (weighted version)

∀δ > 0 ∃c > 0 so that for sufficiently large N,
every $f: \mathbb{Z}_N \rightarrow [0, 1]$ with $\mathbb{E}f \geq \delta$ satisfies

$$AP_3(f) := \mathbb{E}_{x, d \in \mathbb{Z}_N}[f(x)f(x + d)f(x + 2d)] \geq c.$$
Roth's theorem (weighted version)

\( \forall \delta > 0 \ \exists c > 0 \) so that for sufficiently large \( N \), every \( f : \mathbb{Z}_N \rightarrow [0, 1] \) with \( \mathbb{E}f \geq \delta \) satisfies

\[
AP_3(f) := \mathbb{E}_{x, d \in \mathbb{Z}_N} [f(x)f(x + d)f(x + 2d)] \geq c.
\]

Sparse setting: some sparse host set \( S \subset \mathbb{Z}_N \).
More generally, use a normalized measure:

\[
\nu : \mathbb{Z}_N \rightarrow [0, \infty) \quad \text{with} \quad \mathbb{E}\nu = 1.
\]

E.g., \( \nu = \frac{N}{|S|} 1_S \) normalized indicator function.
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E.g., \( \nu = \frac{N}{|S|}1_S \) normalized indicator function.

The subset \( A \subset S \) with \( |A| \geq \delta |S| \) corresponds to

\[ f : \mathbb{Z}_N \rightarrow [0, \infty), \quad \mathbb{E}f \geq \delta \]

and \( f \) majorized by \( \nu \), meaning that \( f(x) \leq \nu(x) \ \forall x \in \mathbb{Z}_N \).
Roth’s theorem (weighted version)

∀δ > 0 ∃c > 0 so that for sufficiently large N,
every $f : \mathbb{Z}_N \rightarrow [0, 1]$ with $\mathbb{E}f \geq \delta$ satisfies $AP_3(f) \geq c$.

Relative Roth theorem (Conlon, Fox, Z.)

∀δ > 0 ∃c > 0 so that for sufficiently large N, if

- $\nu : \mathbb{Z}_N \rightarrow [0, \infty)$ satisfies the 3-linear forms condition, and
- $f : \mathbb{Z}_N \rightarrow [0, \infty)$ majorized by $\nu$ and $\mathbb{E}f \geq \delta$, then

$$AP_3(f) \geq c.$$  

Recall $AP_3(f) = \mathbb{E}_{x,d \in \mathbb{Z}_N}[f(x)f(x + d)f(x + 2d)]$
3-linear forms condition

The density of $K_{2,2,2}$ in $\mathbb{Z}_N 	imes \mathbb{Z}_N 	imes \mathbb{Z}_N$ is

\[ \nu(2x + y) \nu(x - z) \nu(-y - 2z) \]
Relative Roth theorem (Conlon, Fox, Z.)

\( \forall \delta > 0 \exists c > 0 \) so that for sufficiently large \( N \), if

- \( \nu : \mathbb{Z}_N \rightarrow [0, \infty) \) satisfies the 3-linear forms condition, and
- \( f : \mathbb{Z}_N \rightarrow [0, \infty) \) majorized by \( \nu \) and \( \mathbb{E} f \geq \delta \), then

\[ AP_3(f) \geq c. \]

\( \nu : \mathbb{Z}_N \rightarrow [0, \infty) \) satisfies the 3-linear forms condition if

\[
\mathbb{E} \left[ \nu(2x + y)\nu(2x' + y)\nu(2x + y')\nu(2x' + y') \cdot \nu(x - z)\nu(x' - z)\nu(x - z')\nu(x' - z') \cdot \nu(-y - 2z)\nu(-y' - 2z)\nu(-y - 2z')\nu(-y' - 2z') \right] = 1 + o(1)
\]

as well as if any subset of the 12 factors were deleted.
Transference

Start with $f \leq \nu$

(sparse) $f : \mathbb{Z}_N \to [0, \infty)$ $\mathbb{E}f \geq \delta$
Transference

Start with \( f \leq \nu \)

(sparse) \( f : \mathbb{Z}_N \rightarrow [0, \infty) \quad \mathbb{E}f \geq \delta \)

Dense model theorem: one can approximate \( f \) (in cut norm) by

(dense) \( \tilde{f} : \mathbb{Z}_N \rightarrow [0, 1] \quad \mathbb{E}\tilde{f} = \mathbb{E}f \)

Counting lemma implies

\[ \text{AP}_3(f) \approx \text{AP}_3(\tilde{f}) \geq c \quad \text{[By Roth's Thm (weighted version)]} \]

\( \Rightarrow \) relative Roth theorem
Start with $f \leq \nu$

(sparse) $f : \mathbb{Z}_N \to [0, \infty)$ \hspace{1cm} $\mathbb{E}f \geq \delta$

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$\Rightarrow$ relative Roth theorem
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$\implies$ relative Roth theorem
In what sense does $0 \leq \tilde{f} \leq 1$ approximate $0 \leq f \leq \nu$?
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- Previous approach (Green–Tao): Gowers uniformity norm
- Our approach: cut norm (aka discrepancy)
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Using cut norm:

- Cheaper dense model theorem
- Trickier counting lemma
Cut norm for weighted bipartite graph (Frieze-Kannan):

\[ g : X \times Y \rightarrow \mathbb{R} \]

\[ \|g\|_{\square} := \frac{1}{|X||Y|} \sup_{A \subseteq X, B \subseteq Y} \left| \sum_{x \in A, y \in B} g(x, y) \right| \]
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Cut norm for \( \mathbb{Z}_N \): \( f: \mathbb{Z}_N \rightarrow \mathbb{R} \)

\[ \|f\|_{\square} := \frac{1}{N^2} \sup_{A, B \subset \mathbb{Z}_N} \left| \sum_{x \in A \atop y \in B} f(x + y) \right| \]
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\|g\|_\square := \frac{1}{|X||Y|} \sup_{A \subseteq X, B \subseteq Y} \left| \sum_{x \in A, y \in B} g(x, y) \right|
\]

Cut norm for \( \mathbb{Z}_N \):
\[ f : \mathbb{Z}_N \rightarrow \mathbb{R} \]
\[
\|f\|_\square := \frac{1}{N^2} \sup_{A, B \subseteq \mathbb{Z}_N} \left| \sum_{x \in A, y \in B} f(x + y) \right|
\]

Dense model theorem
Assume \( \nu : \mathbb{Z}_N \rightarrow [0, \infty) \) satisfies \( \|\nu - 1\|_\square = o(1) \).
Then \( \forall \ 0 \leq f \leq \nu, \exists \tilde{f} : \mathbb{Z}_N \rightarrow [0, 1] \) s.t. \( \|f - \tilde{f}\|_\square = o(1) \).
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Assume $\nu: \mathbb{Z}_N \to [0, \infty)$ satisfies $\|\nu - 1\|_{\square} = o(1)$.
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Proof of the general dense model theorem
1. Regularity-type energy-increment argument (Green–Tao, Tao–Ziegler)
2. Separating hyperplane theorem / minimax theorem + Weierstrass polynomial approximation theorem (Gowers & Reingold–Trevisan–Tulsiani–Vadhan)
Specialized/simplified for the cut norm on $\mathbb{Z}_N$. 

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Specialized/simplified for the cut norm on $\mathbb{Z}_N$ (Z.)
Higher cut norms

For 4-AP

3-uniform weighted hypergraph $g : X \times Y \times Z \rightarrow \mathbb{R}$, define

$$\|g\|_{\Box} = \frac{1}{|X||Y||Z|} \sup_{A \subset Y \times Z} \sup_{B \subset X \times Z} \sup_{C \subset X \times Y} \left| \sum_{\substack{x \in X, y \in Y, z \in Z \ (y,z) \in A \ (x,z) \in B \ (x,y) \in C}} g(x, y, z) \right|.$$ 

i.e., supremum taken over all 2-graphs between $X, Y, Z$
Transference

Start with $f \leq \nu$

\[(\text{sparse}) \quad f : \mathbb{Z}_N \to [0, \infty) \quad \mathbb{E}f \geq \delta]\]

**Dense model theorem**: one can approximate $f$ (in cut norm) by

\[(\text{dense}) \quad \tilde{f} : \mathbb{Z}_N \to [0, 1] \quad \mathbb{E}\tilde{f} = \mathbb{E}f\]

**Counting lemma** implies

\[AP_3(f) \approx AP_3(\tilde{f}) \geq c \quad [\text{By Roth’s Thm (weighted version)}]\]

$\implies$ relative Roth theorem
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Counting lemma

Weighted graphs $g, \tilde{g} : (X \times Y) \cup (X \times Z) \cup (Y \times Z) \to \mathbb{R}$

Triangle density $t(g) := \mathbb{E}_{x,y,z}[g(x,y)g(x,z)g(y,z)]$

Triangle counting lemma (dense setting)

Assume $0 \leq g, \tilde{g} \leq 1$. If $\|g - \tilde{g}\|_\square \leq \epsilon$, then

$$t(g) = t(\tilde{g}) + O(\epsilon).$$
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This argument doesn't work in the sparse setting ($g$ unbounded)
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**Sparse triangle counting lemma (Conlon, Fox, Z.)**

Assume that $\nu$ satisfies the 3-linear forms condition. If $0 \leq g \leq \nu$, $0 \leq \tilde{g} \leq 1$ and $\|g - \tilde{g}\|_{\Box} = o(1)$, then

$$t(g) = t(\tilde{g}) + o(1)$$

Recall $t(g) = \mathbb{E}[g(x, y)g(x, z)g(y, z)]$
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Proof ingredients

1. Cauchy-Schwarz
2. Densification
3. Apply cut norm/discrepancy (as in dense case)
Densification

\[ \mathbb{E}[g(x, z)g(y, z)g(x, z')g(y, z')] \]

Made \( X \times Y \) dense. Now repeat for \( X \times Z \) & \( Y \times Z \).

Reduce to dense setting.
Densification

\[ \mathbb{E}[g(x, z)g(y, z)g(x, z')g(y, z')] \]

Set \( g'(x, y) := \mathbb{E}_{z'}[g(x, z')g(y, z')] \), i.e., codegrees

\( g'(x, y) \lesssim 1 \) for almost all \((x, y)\)
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Coming Soon

The Green-Tao theorem: an exposition
A gentle exposition giving a complete & self-contained proof of the Green-Tao theorem (assuming Szemerédi’s theorem)

~ 25 pages

THANK YOU!