Algorithms in invariant theory

Visu Makam
(joint work with Harm Derksen)

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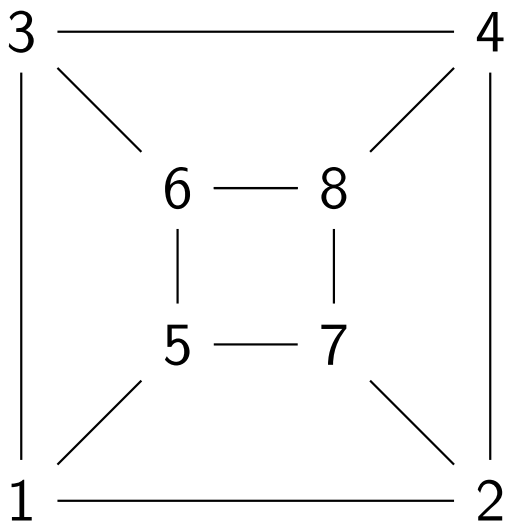
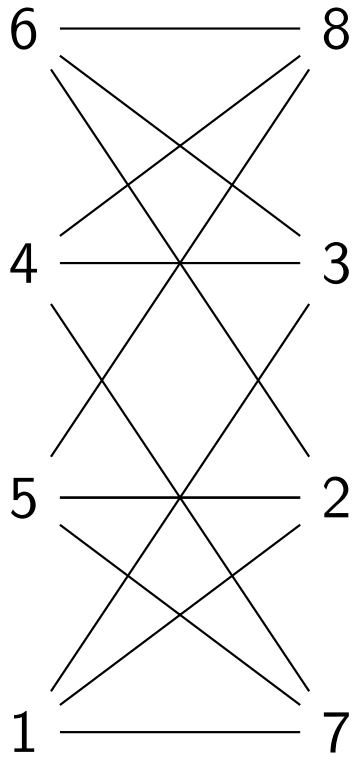
Graph isomorphism

- Are the two graphs isomorphic?





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where M_σ is the permutation matrix associated to σ.

- $G_1 \cong G_2$ if and only if A_{G_1} and A_{G_2} are in the same orbit for the action of S_n.

Polynomial functions on vector spaces

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- The ring of polynomial functions is $K[x_1, \ldots, x_n]$.

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- This is a graded (by degree) subring of $K[x_1, ..., x_n]$.

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- **Fact:** For finite group actions, orbits can be distinguished by invariant polynomials.
- **Recall:** Invariant rings are finitely generated.
- Algorithm for Graph isomorphism: Find a finite set of generators for $K[x_{i,j} | 1 \leq i, j \leq n]^{S_n}$, and test all.

An example

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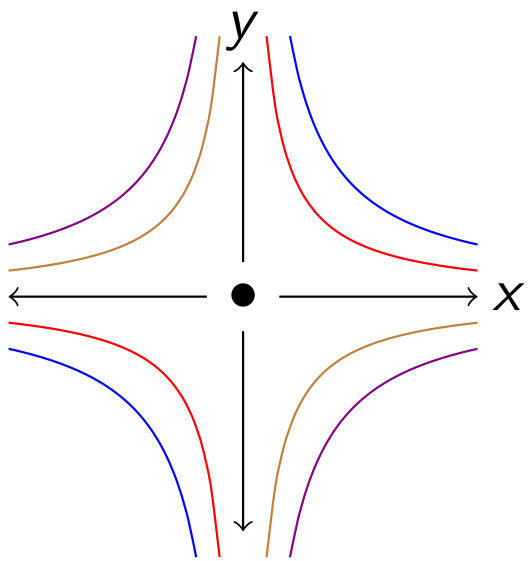

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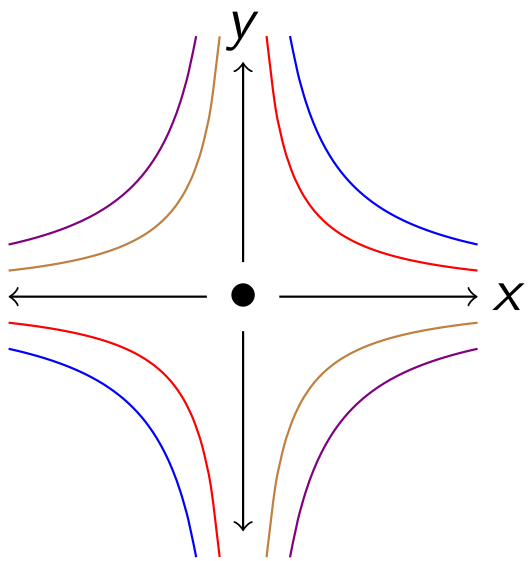

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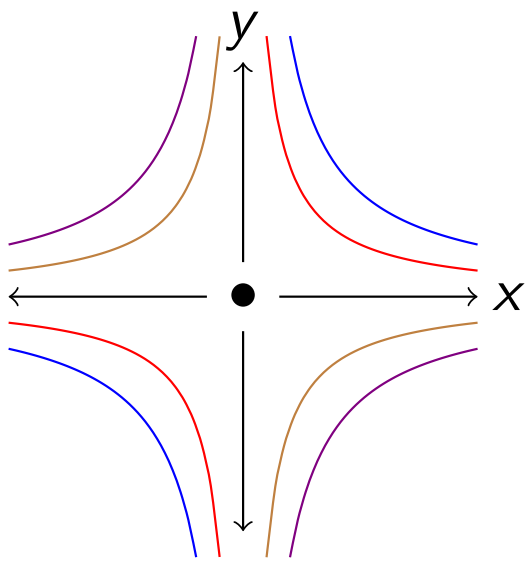

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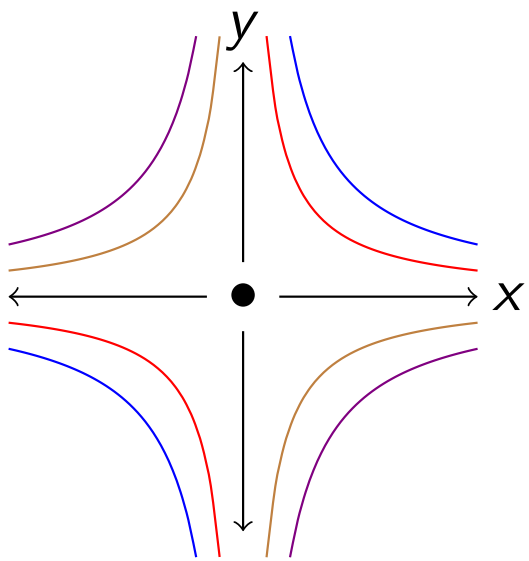

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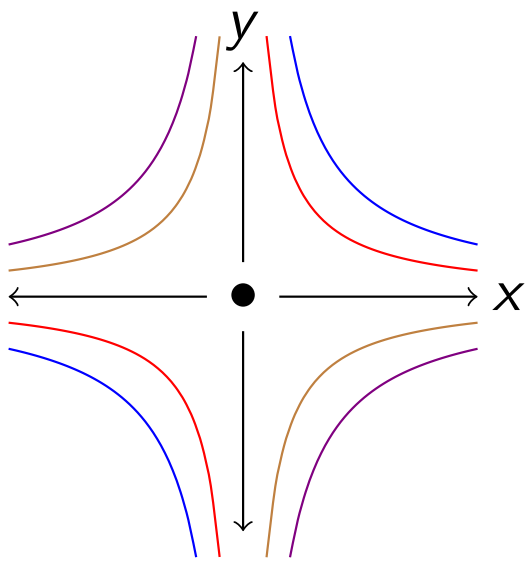

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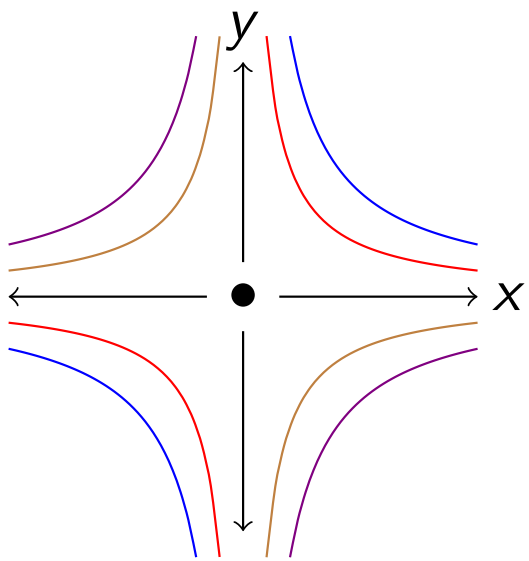

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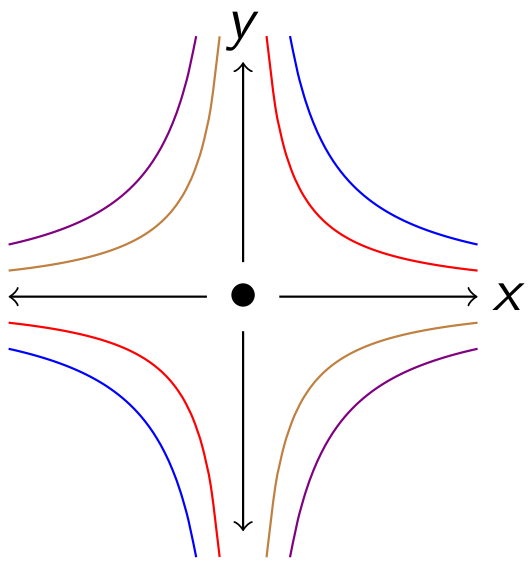

- What are the orbits?
  1. $xy = k$ for $k \neq 0$;
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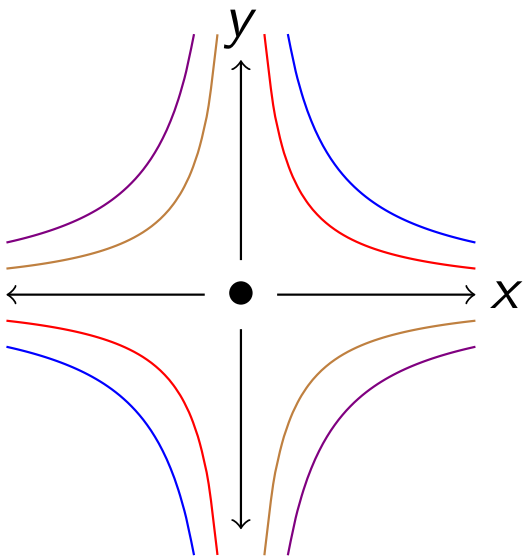
- Can invariant polynomials distinguish between orbits?

Example continued



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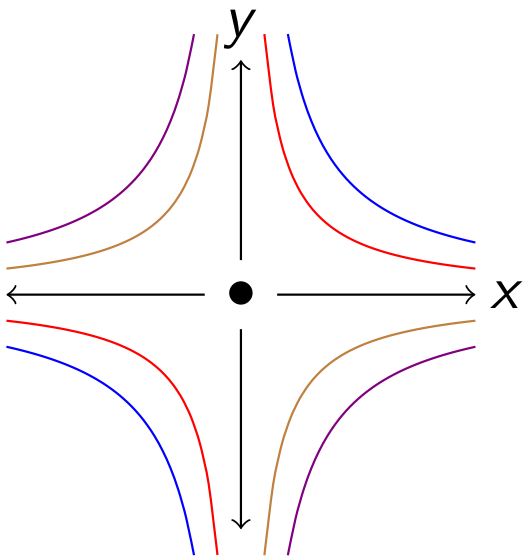

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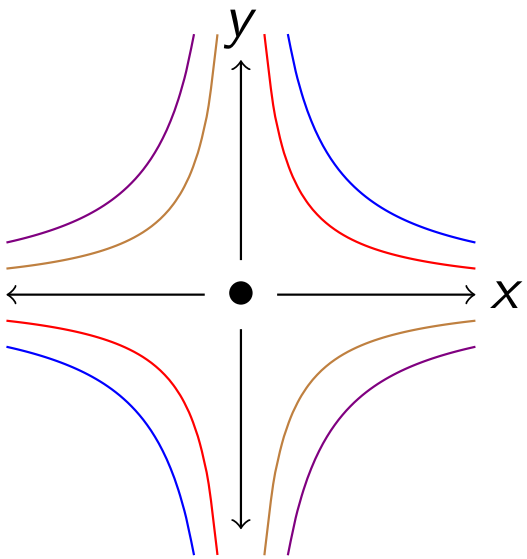

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- Their orbit closures intersect!

Orbit closure problem

Problem (Orbit closure problem)

$G \curvearrowright V$. Decide if orbit closures of v and w intersect.

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- All classical groups are reductive - GL_n, SL_n, Sp_n, O_n, tori, finite groups etc.

Null cone

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- A special case of the orbit closure problem!

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- **Main problems:** degree bounds, null cone and orbit closure.
- **Main objects:** Matrix invariants and matrix semi-invariants.

Matrix semi-invariants

- Consider the left-right action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on $\mathrm{Mat}_{n,n}^m$ given by:

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- How about $\det \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix}$?

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- No homogeneous invariants of degree k unless $n|k$.

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- There is no shrunk subspace.

Results on matrix semi-invariants

Theorem (Derksen, Makam)

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- Randomized algorithm for null cone and orbit closure problems!

Derksen's bound in perspective

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- GGOW, IQS give deterministic algorithm for null cone for matrix semi-invariants.
- Is there a deterministic algorithm for the orbit closure problem for matrix semi-invariants?

How about deterministic algorithms?

- Deterministic algorithm for null cone $\overset{?}{\approx}$ Deterministic algorithm for orbit closure

- GGOW, IQS give deterministic algorithm for null cone for matrix semi-invariants.
- Is there a deterministic algorithm for the orbit closure problem for matrix semi-invariants?
- We will need to look at matrix invariants first.

Matrix invariants

- Consider the simultaneous conjugation action of GL_n on $\mathrm{Mat}_{n,n}^m$ given by

$$g \cdot (X_1, \ldots, X_m) = (gX_1g^{-1}, \ldots, gX_mg^{-1}).$$

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- Observe that $\mathrm{Tr}(X_w)$ is an invariant polynomial (of deg k).

Matrix invariants

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- In other words, a degree bound of n^2.
- However, this is still an exponentially large generating set!

Orbit closure problem for matrix invariants

- Two algorithms.

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- **Raz-Shpilka:** PIT for non-commutative polynomials (ROABPs) is in polynomial time.

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- **Key idea:** Can extract in polynomial time a subset of this spanning set which forms a basis.

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- Algorithm giving an algebraic result!

Matrix semi-invariants revisited

- Recall the left-right action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on m-tuples of $n \times n$ matrices

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- Equivalently, want to know if $f_T(A) \neq f_T(B)$ for some T.

Orbit closure for matrix semi-invariants

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- From now on, assume w.l.o.g that A and B are not in the null cone.

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  Let's leave that as an exercise!

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- **Upshot:** Suffices to have an invertible matrix in the span of the A_i's.

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- **Derksen, Makam:** Get a bound of degree $2n^{3.5}$ for separating invariants (vs degree bound of n^6).
- A very different (analytic) algorithm in Yuanzhi Li's talk tomorrow!

Recap and further directions

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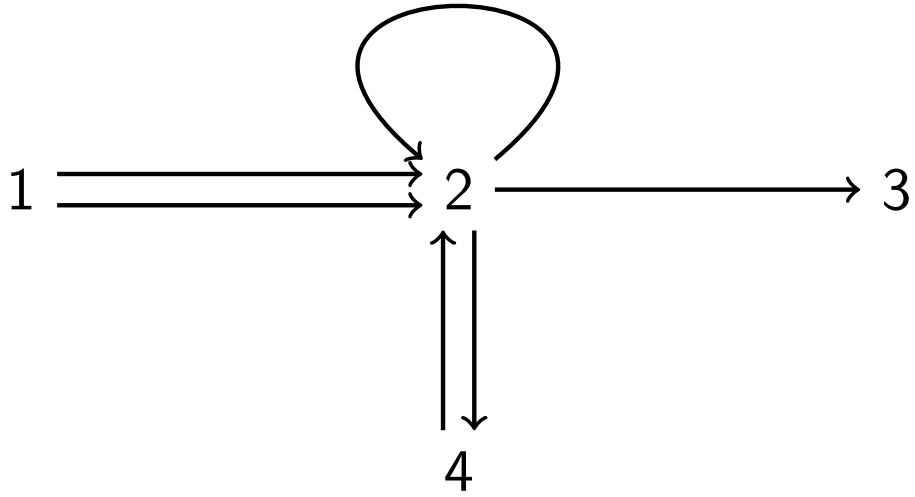
  2. Positive characteristic.

Quivers

- A quiver is a directed graph.



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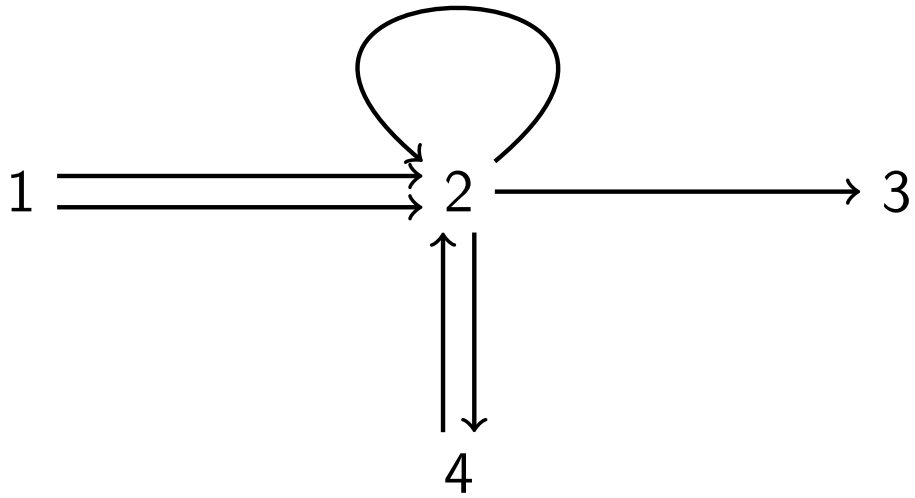

- In general can have directed cycles and multiple edges.

Quiver representation

- A quiver representation is a the data of a vector space at each vertex, and linear maps for each arrow.

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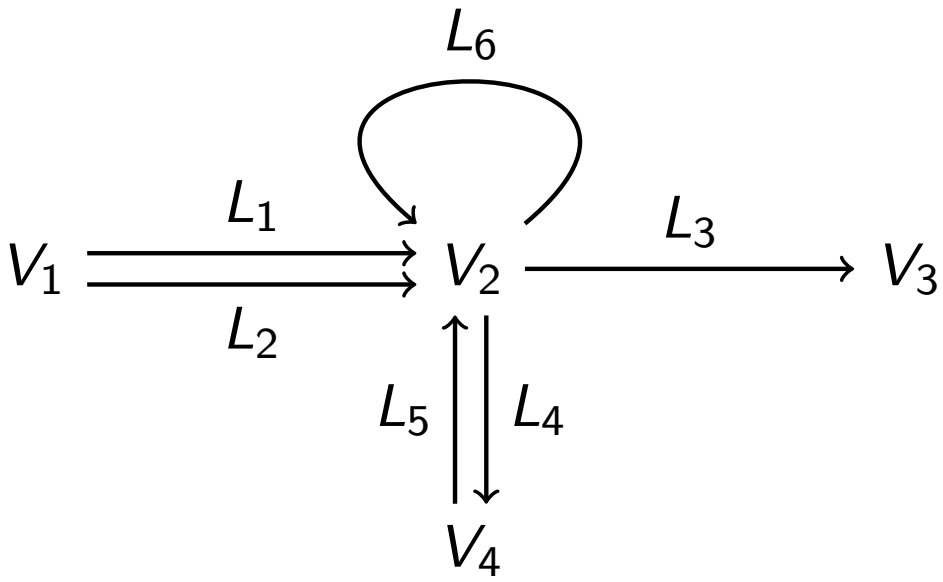

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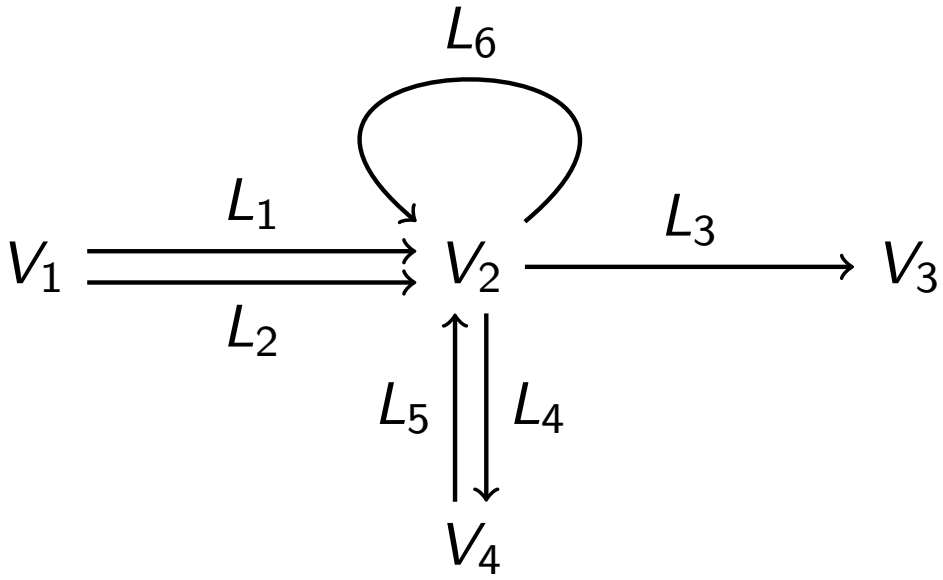

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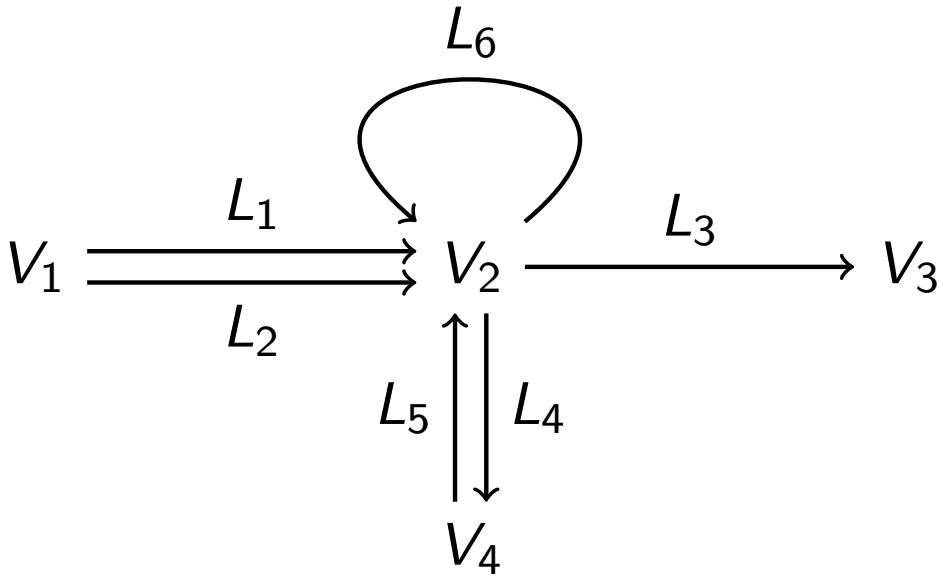

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- Might as well choose the vector spaces to be $K^{\dim(V_i)}$

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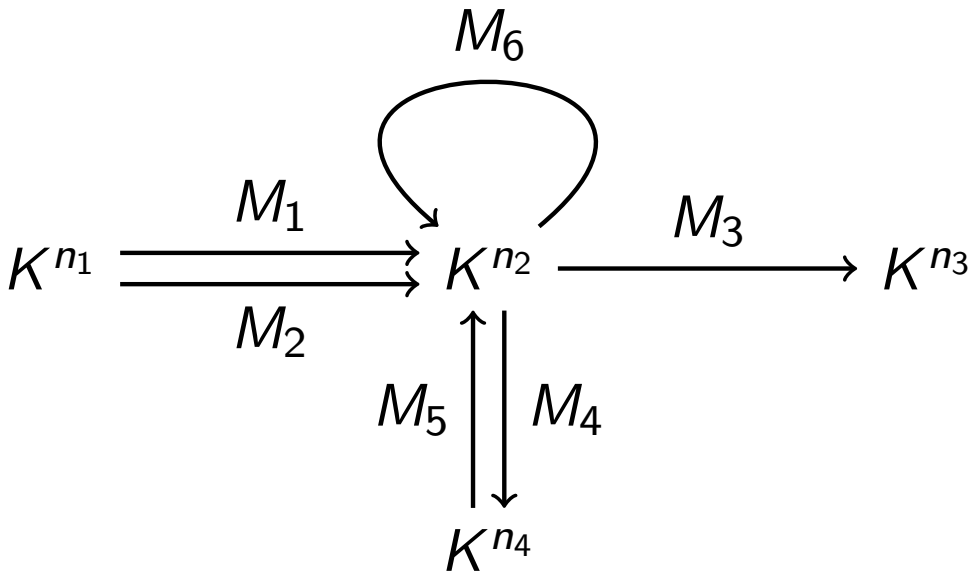

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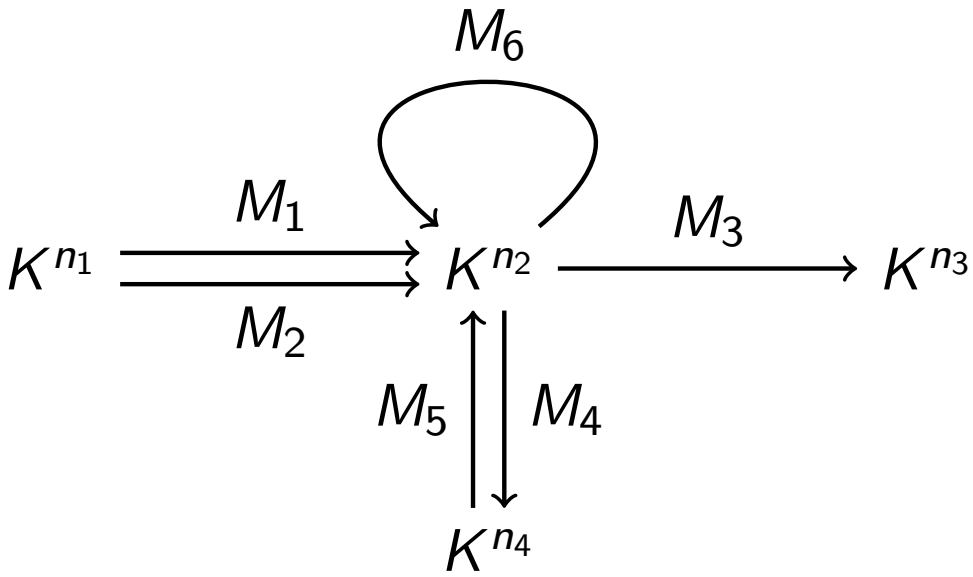

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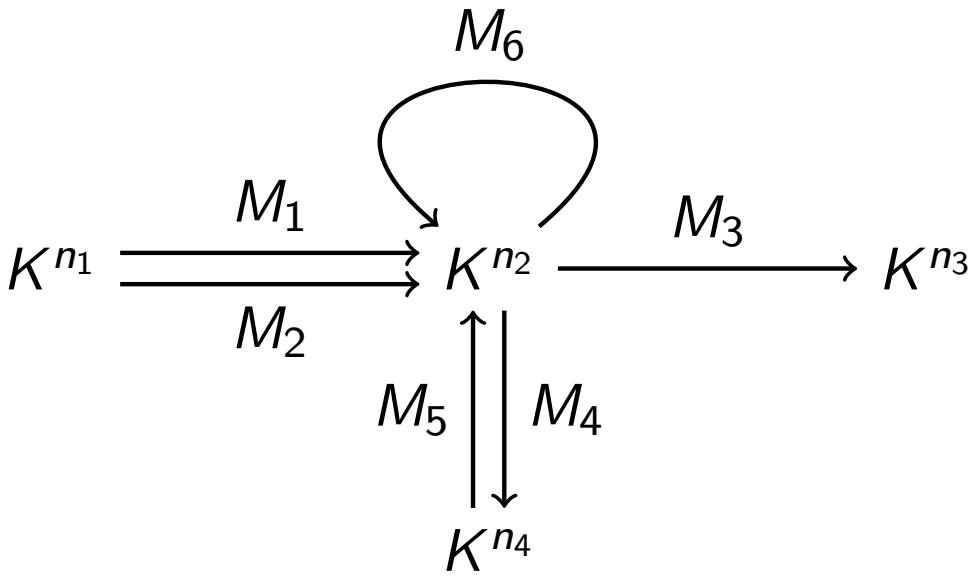

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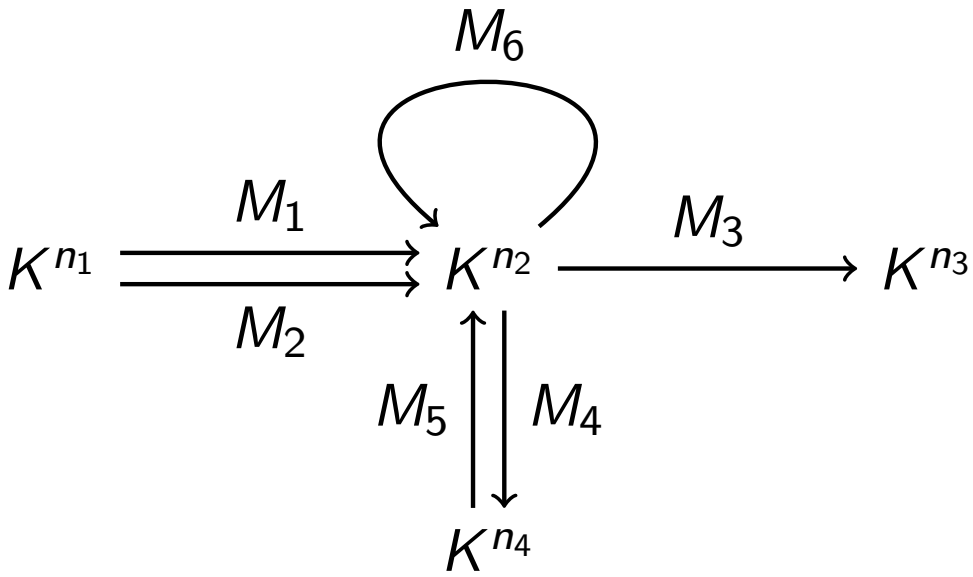

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- Precisely when they are related by base changes (at the vertices).

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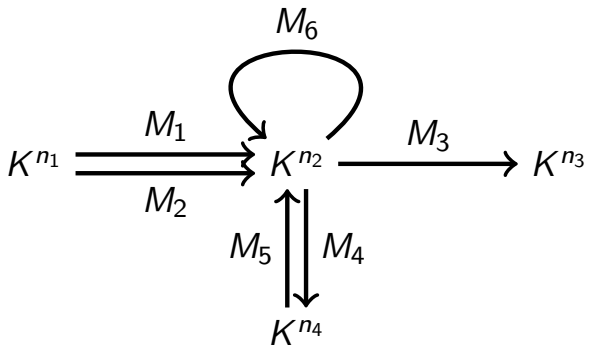

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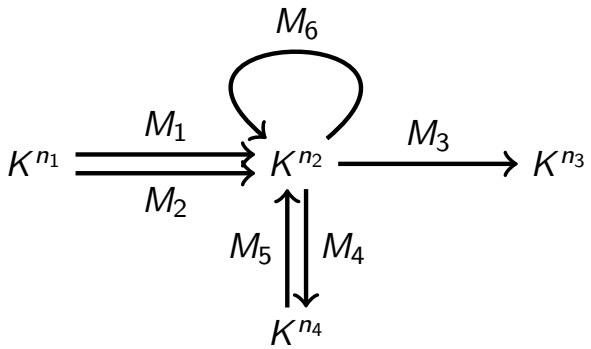

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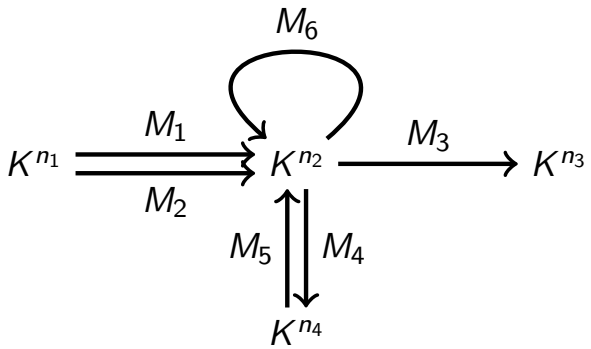

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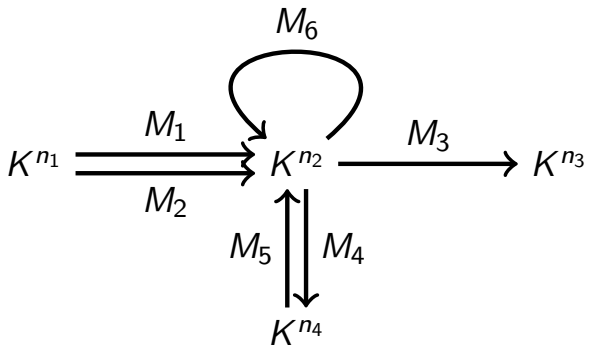

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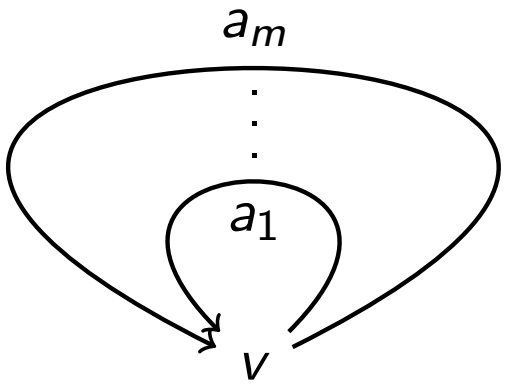
- Consider m-loop quiver:



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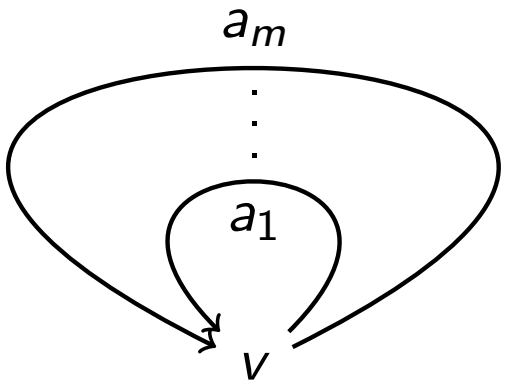

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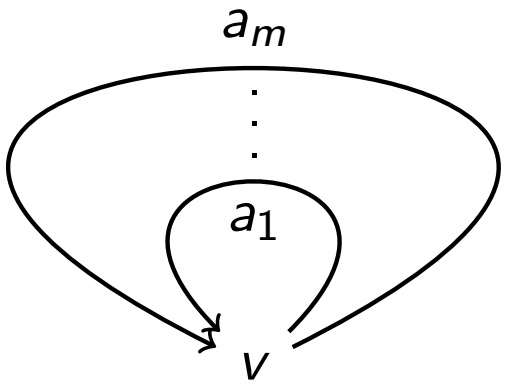

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- This is precisely the action of GL_n by simultaneous conjugation.

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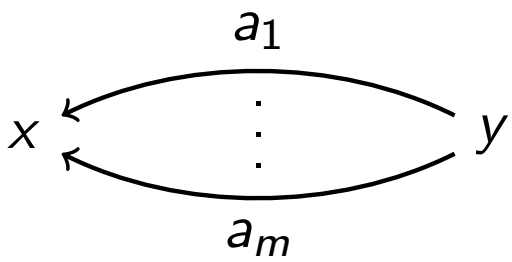

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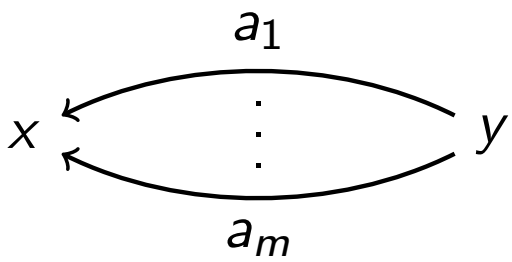

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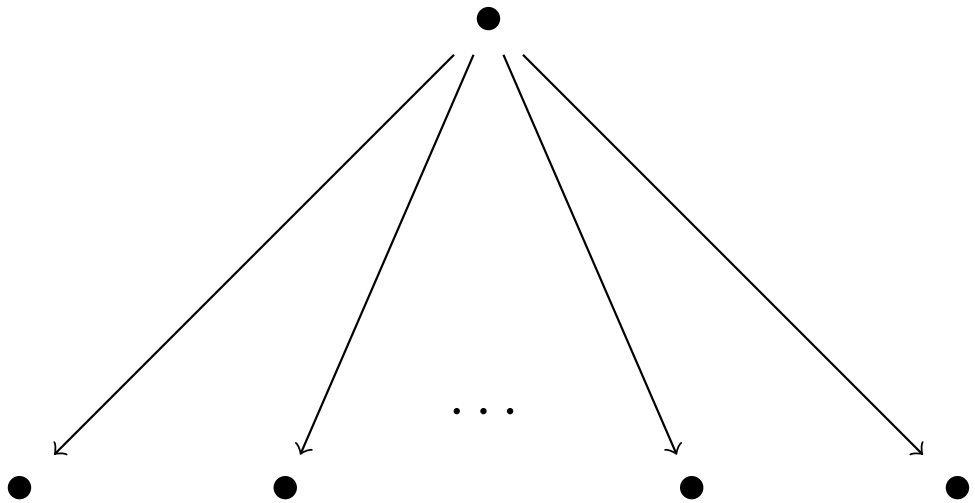
- Consider the action of $\mathrm{SL}_p \times \mathrm{SL}_q$ via base change, this is the left-right action!

The quiver for Brascamp-Lieb inequalities

The quiver that governs the results on Brascamp-Lieb inequalites is the star quiver



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- Representation theory is more complicated.
- Commutative algebra statements do not always go through.

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- **Donkin:** The set $\sigma_i(A_w)$ is a generating set of invariants (highly non-trivial!)

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- Curiously, the bound on null cone for matrix semi-invariants holds in all characteristics!

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