Algorithms in invariant theory

Visu Makam
(joint work with Harm Derksen)

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Are the two graphs isomorphic?
Graph isomorphism via group actions

- Fix a vertex set \( \{1, 2, \ldots, n\} \).

  Graph \( G \leftrightarrow \) adjacency matrix \( A_G \in \text{Mat}_{n,n} \).
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- \( S_n \circlearrowright \text{Mat}_{n,n} \) by conjugation, i.e.,

  \[
  \sigma \cdot A = M_\sigma A M_\sigma^{-1},
  \]

  where \( M_\sigma \) is the permutation matrix associated to \( \sigma \).
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- $S_n \circ \text{Mat}_{n,n}$ by conjugation, i.e.,
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  \sigma \cdot A = M_{\sigma} A M_{\sigma}^{-1},
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  where $M_{\sigma}$ is the permutation matrix associated to $\sigma$.

- $G_1 \cong G_2$ if and only if $A_{G_1}$ and $A_{G_2}$ are in the same orbit for the action of $S_n$. 
Let $V = K^n$ be a vector space.
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Let $e_1, \ldots, e_n$ denote the standard basis.
Polynomial functions on vector spaces

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The ring of polynomial functions is $K[x_1, \ldots, x_n]$. 
Invariant polynomials

Suppose we have a group \( G \) acting on \( V \) by linear transformations.
Invariant polynomials

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- This is a graded (by degree) subring of $K[x_1, \ldots, x_n]$. 
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Graph isomorphism revisited

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- Suppose \( f \) is an invariant polynomial such that \( f(A_{G_1}) \neq f(A_{G_2}) \), then \( G_1 \not\cong G_2 \).
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- **Fact:** For finite group actions, orbits can be distinguished by invariant polynomials.
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**Recall:** Invariant rings are finitely generated.

**Algorithm for Graph isomorphism:** Find a finite set of generators for $\mathbb{K}[x_{i,j}| 1 \leq i, j \leq n]^{S_n}$, and test all.
An example

- Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ by scaling, i.e., $t \cdot (a, b) = (ta, t^{-1}b)$.

  - What are the orbits?
    1. $xy = k$ for $k \neq 0$;
    2. $x$-axis minus the origin;
    3. $y$-axis minus the origin;
    4. The origin.
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- The invariant ring is $K[xy]$. 

Can't distinguish between the three orbits: $x$-axis minus origin, $y$-axis minus origin and the origin. Why? Their orbit closures intersect!
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- The invariant ring is $K[xy]$.
- Can’t distinguish between the three orbits: x-axis minus origin, y-axis minus origin and the origin. Why?
- Their orbit closures intersect!
Problem (Orbit closure problem)

$G \bowtie V$. Decide if orbit closures of $v$ and $w$ intersect.

Theorem

The orbit closures of two points intersect if and only if they cannot be distinguished by an invariant polynomial.

Above theorem requires group to be reductive.

All classical groups are reductive - $GL_n$, $SL_n$, $Sp_n$, $O_n$, tori, finite groups etc.
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Null cone

Problem (Null cone membership problem)

Decide if a given point is in the null cone.
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  \text{null cone} = \{ v \in V \mid 0 \in G \cdot v \}.
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- A special case of the orbit closure problem!
If life was easy...

- **Best case scenario:** We have a small set of (homogeneous) generators $f_1, \ldots, f_r$ that are efficient to compute.

Algorithm for null cone: Check if $f_i(v) \neq 0$ for some $i$.

Algorithm for orbit closure: Check if $f_i(v) \neq f_i(w)$ for some $i$. How can we get algorithms if we are not in the best case scenario? Start with degree bounds.

Main problems: degree bounds, null cone and orbit closure.

Main objects: Matrix invariants and matrix semi-invariants.
Null cone and orbit closure problems

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- **Main problems:** degree bounds, null cone and orbit closure.

- **Main objects:** Matrix invariants and matrix semi-invariants.
Consider the left-right action of $\text{SL}_n \times \text{SL}_n$ on $\text{Mat}_{n,n}^m$ given by:

$$(P, Q) \cdot (X_1, \ldots, X_m) = (PX_1Q^{-1}, \ldots, PX_mQ^{-1}).$$
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- $\det(X_1)$ is an invariant polynomial.

- $\det(\sum_i c_i X_i)$ is also an invariant polynomial.

- How about $\det(X_1 X_2 X_2 X_3)$?
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- Are these all?
Matrix semi-invariants

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- How about $\det \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix}$?
Description of invariants

For $T = (T_1, \ldots, T_m) \in \text{Mat}_{d,d}^m$, we define an invariant $f_T$ by
Matrix semi-invariants

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For $T = (T_1, \ldots, T_m) \in \text{Mat}^m_{d, d}$, we define an invariant $f_T$ by

$$f_T(X_1, \ldots, X_m) = \det(X_1 \otimes T_1 + \cdots + X_m \otimes T_m).$$
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- $f_T$ is a homogeneous invariant of degree $dn$. 

\[ \text{Theorem (Derksen-Weyman, Domokos-Zubkov, Schofield-Van den Bergh)} \]

The linear span of \{ $f_T | T \in \text{Mat}_{d,d}^m$, $d$ \} gives all degree $dn$ invariants.

No homogeneous invariants of degree $k$ unless $n | k$. 

Visu Makam (joint work with Harm Derksen) 

Algorithms in invariant theory 

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Matrix semi-invariants

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- No homogeneous invariants of degree $k$ unless $n|k$. 
Characterizations of the null cone

The following are equivalent (Recall from K.V.’s talk):

$$(A_1, \ldots, A_m) \text{ not in null cone.}$$

$$f(T(A_1, \ldots, A_m)) = \det(A_1 \otimes T_1 + \cdots + A_m \otimes T_m) \neq 0 \text{ for some } T.$$
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Some blow-up contains an invertible matrix.

$\text{ncrk}(t_1 A_1 + \cdots + t_m A_m)$ is full.

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Results on matrix semi-invariants

Theorem (Derksen, Makam)

For every $d \geq n - 1$, $(A_1, \ldots, A_m)$ is not in null cone if and only if there exists $T \in \text{Mat}_{d,d}^m$ such that $f_T(A) \neq 0$. 

Equivalently, there is an invertible matrix in the $d$th blow-up.

IQS algorithm = constructive version of above result.

Invariants of polynomial degree suffice to define the null cone.

Corollary (Derksen, Makam)

We have an upper bound of $n^6$ for the degree of generators for matrix semi-invariants.

Randomized algorithm for null cone and orbit closure problems!
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- Randomized algorithm for null cone and orbit closure problems!
Derksen’s bound in perspective

- Bound for null cone $\approx$ Randomized algorithm for null cone.
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Theorem (Derksen)

Degree bound for generators = Poly(bound for null cone).
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**Theorem (Derksen)**

*Degree bound for generators = Poly(bound for null cone).*
How about deterministic algorithms?

Deterministic algorithm for null cone

≈

Deterministic algorithm for orbit closure
How about deterministic algorithms?

- Deterministic algorithm for null cone ≈ Deterministic algorithm for orbit closure

- GGOW, IQS give deterministic algorithm for null cone for matrix semi-invariants.
How about deterministic algorithms?

- Deterministic algorithm for null cone

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- GGOW, IQS give deterministic algorithm for null cone for matrix semi-invariants.

- Is there a deterministic algorithm for the orbit closure problem for matrix semi-invariants?
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- Deterministic algorithm for null cone \( \approx \) Deterministic algorithm for orbit closure

- GGOW, IQS give deterministic algorithm for null cone for matrix semi-invariants.

- Is there a deterministic algorithm for the orbit closure problem for matrix semi-invariants?

- We will need to look at matrix invariants first.
Consider the simultaneous conjugation action of $\text{GL}_n$ on $\text{Mat}_{n,n}^m$ given by
\[ g \cdot (X_1, \ldots, X_m) = (gX_1g^{-1}, \ldots, gX_mg^{-1}). \]
Matrix invariants

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  \[ g \cdot (X_1, \ldots, X_m) = (gX_1g^{-1}, \ldots, gX_mg^{-1}) \].

- $\text{Tr}(X_1), \text{Tr}(X_1X_2X_3), \text{Tr}(X_1X_3X_2)$ are all invariant polynomials.
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- For a word $w = i_1i_2\ldots i_k$ with $i_j \in \{1, 2, \ldots, m\}$, define $X_w = X_{i_1}X_{i_2}\ldots X_{i_k}$. 
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- For a word $w = i_1i_2\ldots i_k$ with $i_j \in \{1, 2, \ldots, m\}$, define $X_w = X_{i_1}X_{i_2}\ldots X_{i_k}$.
- Observe that $Tr(X_w)$ is an invariant polynomial (of deg $k$).
Matrix invariants

Theorem (Procesi/Sibirskii)

Invariants of the form $\text{Tr}(X_w)$ form an (infinite) generating set (in char 0).
Matrix invariants

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- In other words, a degree bound of $n^2$.
- However, this is still an exponentially large generating set!
Orbit closure problem for matrix invariants

- Two algorithms.

Algorithm 1: (Forbes, Shpilka, Mulmuley's construction)
Given $A = (A_1, \ldots, A_m) \in \text{Mat}_{m \times n}$, construct a (non-commutative) polynomial $P_d(A)$. The coefficient of any monomial in $P_d(A)$ is of the form $\text{Tr}(A_{i_1} A_{i_2} \cdots A_{i_d})$ (and all words of length $d$ appear).

Check if $P_d(A) - P_d(B) = 0$ for $1 \leq d \leq n^2$.

Raz-Shpilka: PIT for non-commutative polynomials (ROABPs) is in polynomial time.
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Our algorithm

- **Algorithm 2:** (Derksen, Makam).

\[ \text{Given } (A_1, \ldots, A_m) \text{ and } (B_1, \ldots, B_m), \text{ construct for all } i, C_i = (A_i 0 0 B_i). \]

Consider the algebra \( C \subset \text{Mat}_{2n, 2n} \) generated by \( C_1, \ldots, C_m \).

\( C \) is spanned by words \( C_w = C_{i_1} C_{i_2} \cdots C_{i_k} = (A_{i_1} \cdots A_{i_k} 0 0 B_{i_1} \cdots B_{i_k}) \).

**Key idea:** Can extract in polynomial time a subset of this spanning set which forms a basis.

Visu Makam (joint work with Harm Derksen)

Algorithms in invariant theory

June 6, 2018 22 / 46
Our algorithm

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Some observations

- The size of the basis is at most $2n^2$. And we don’t need the degree bound!
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- Algorithm giving an algebraic result!
Matrix semi-invariants revisited

Recall the left-right action of $\text{SL}_n \times \text{SL}_n$ on $m$-tuples of $n \times n$ matrices

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- Equivalently, want to know if $f_T(A) \neq f_T(B)$ for some $T$. 

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Orbit closure algorithms

Orbit closure for matrix semi-invariants

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From now on, assume w.l.o.g that $A$ and $B$ are not in the null cone.
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- Either $\det(B_1) = 1$ or their orbit closures do not intersect.
- If $\det(B_1) = 1$, then can find $(P, Q) \in \text{SL}_n \times \text{SL}_n$ s.t $PB_1Q^{-1} = I$. 

Suffices to check if orbit closures of $(A_2, \ldots, A_m)$ and $(\tilde{B}_2, \ldots, \tilde{B}_m)$ intersect for the simultaneous conjugation action! Let’s leave that as an exercise!
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Replace $B$ by $\tilde{B} = (P, Q) \cdot B = (I, \tilde{B}_2, \ldots, \tilde{B}_m)$. 
Orbit closure for matrix semi-invariants continued

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There is an action of $GL_m$ on $m$-tuples of matrices. For 

$$H = \begin{pmatrix} h_{11} & \cdots & h_{1m} \\ \vdots & \ddots & \vdots \\ h_{m1} & \cdots & h_{mm} \end{pmatrix} \in GL_m,$$

we have 

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Orbit closure for matrix semi-invariants continued

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**Upshot:** Suffices to have an invertible matrix in the span of the \( A_i \)’s.
Summary so far:

1. If $A$ or $B$ are in the null cone, IQS algorithm suffices.
2. If we can find an invertible matrix in the span of the $A_i$'s, then we can detect orbit closure intersection.

Two issues:

1. Even if such an invertible matrix exists, how do you find one efficiently?
2. Such an invertible matrix doesn't always exist.

Key idea: You can find an invertible matrix in a blow-up efficiently!

Derksen, Makam: Get a bound of degree $2n^3$ for separating invariants (vs degree bound of $n^6$).

A very different (analytic) algorithm in Yuanzhi Li's talk tomorrow!
Orbit closure for matrix semi-invariants continued

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  1. Generalize to quivers.
  2. Positive characteristic.
A quiver is a directed graph.
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- In general can have directed cycles and multiple edges.
A quiver representation is the data of a vector space at each vertex, and linear maps for each arrow.
Quiver representation

A quiver representation is a the data of a vector space at each vertex, and linear maps for each arrow.

\[ V_1 \quad L_1 \quad L_2 \quad V_2 \quad L_3 \quad V_3 \]

\[ V_1 \quad L_4 \quad L_5 \quad V_4 \]

Dimension vector is \((\dim(V_1), \dim(V_2), \dim(V_3), \dim(V_4))\).
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Might as well choose the vector spaces to be \(K^{\dim(V_i)}\).
Quiver representation: Take 2

- Fix standard vector spaces at the vertices.
Quiver representation: Take 2

- Fix standard vector spaces at the vertices.

\[ \begin{aligned}
K^{n_1} & \xrightarrow{M_1} & K^{n_2} & \xrightarrow{M_3} & K^{n_3} \\
M_2 & \quad & & \\
K^{n_4} & \xleftarrow{M_4} & K^{n_2} & \xleftarrow{M_5} & K^{n_4} \\
M_6 & \quad & 
\end{aligned} \]
Quiver representation: Take 2

- Fix standard vector spaces at the vertices.

\[ M_i \text{'s are now matrices of the appropriate size!} \]
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\[ K^{n_1} \xrightarrow{M_1} K^{n_2} \xrightarrow{M_3} K^{n_3} \]

\[ K^{n_2} \xrightarrow{M_6} K^{n_4} \]

- \( M_i \)’s are now matrices of the appropriate size!
- When are two representations isomorphic?
Quiver representation: Take 2

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\[ K^{n_1} \xrightarrow{M_1} K^{n_2} \xrightarrow{M_3} K^{n_3} \]

\[ K^{n_2} \xrightarrow{M_6} K^{n_1} \]

\[ K^{n_4} \]

- \( M_i \)'s are now matrices of the appropriate size!
- When are two representations isomorphic?
- Precisely when they are related by base changes (at the vertices).
Quiver representation: Take 2

- Fix a dimension vector $n = (n_1, n_2, n_3, n_4)$. 
Quiver representation: Take 2

- Fix a dimension vector $\mathbf{n} = (n_1, n_2, n_3, n_4)$.
Fix a dimension vector \( \underline{n} = (n_1, n_2, n_3, n_4) \).

An \( \underline{n} \)-dimensional representation is given by a point in

\[
\text{Rep}(Q, \underline{n}) = \text{Mat}^{\oplus 2}_{n_2, n_1} \oplus \text{Mat}_{n_3, n_2} \oplus \text{Mat}_{n_4, n_2} \oplus \text{Mat}_{n_2, n_4} \oplus \text{Mat}_{n_2, n_2}
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Base change group is \( \text{GL}(\underline{n}) = \text{GL}_{n_1} \times \text{GL}_{n_2} \times \text{GL}_{n_3} \times \text{GL}_{n_4} \).
Quiver representation: Take 2

- Fix a dimension vector \( \underline{n} = (n_1, n_2, n_3, n_4) \).

\[
\begin{array}{c}
\text{\( K^{n_1} \)} \\
\downarrow M_1 \quad \downarrow M_2 \quad \uparrow M_3 \\
\text{\( K^{n_2} \)} \\
\downarrow M_4 \quad \uparrow M_5 \\
\text{\( K^{n_4} \)} \\
\text{\( K^{n_3} \)} \\
\end{array}
\]

- An \( n \)-dimensional representation is given by a point in

\[
\text{Rep}(Q, \underline{n}) = \text{Mat}_{n_2,n_1}^{\oplus 2} \oplus \text{Mat}_{n_3,n_2} \oplus \text{Mat}_{n_4,n_2} \oplus \text{Mat}_{n_2,n_4} \oplus \text{Mat}_{n_2,n_2}
\]

- Base change group is \( \text{GL}(\underline{n}) = \text{GL}_{n_1} \times \text{GL}_{n_2} \times \text{GL}_{n_3} \times \text{GL}_{n_4} \).

\[
\text{Rep}(Q, \underline{n})/ \text{GL}(\underline{n}) = \text{isomorphism classes of } n\text{-dim’l reps.}
\]
The loop quiver

- Consider $m$-loop quiver:

- If you fix the vector space $K^n$ at the vertex, then a representation is just an $m$-tuple of $n \times n$ matrices.
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This is precisely the action of $\text{GL}_n$ by simultaneous conjugation.
The Kronecker quiver

- Consider the $m$-Kronecker quiver:

\[
\begin{array}{c}
\vdots \\
x \\
\vdots \\
y
\end{array}
\quad
\begin{array}{c}
a_1 \\
\vdots \\
am \\
\end{array}
\]

- Fixing vector spaces $K^p$ and $K^q$, a representation is the data of an $m$-tuple of $p \times q$ matrices.
The Kronecker quiver

- Consider the $m$-Kronecker quiver:

```
  x ← a_1 ← y
  .   .
  .   .
  .   .
  x ← a_m ← y
```

- Fixing vector spaces $K^p$ and $K^q$, a representation is the data of an $m$-tuple of $p \times q$ matrices.

- Consider the action of $\text{SL}_p \times \text{SL}_q$ via base change, this is the left-right action!
The quiver for Brascamp-Lieb inequalities

The quiver that governs the results on Brascamp-Lieb inequalities is the star quiver

\[ 
\begin{array}{c}
\bullet \\
\uparrow \\
\bullet \\
\uparrow \\
\bullet \\
\uparrow \\
\bullet \\
\uparrow \\
\bullet \\
\end{array}
\]
Analytic algorithms cannot be adapted! So, you have to deal with the algebra.
Problems in positive characteristic

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- Description of invariants is harder to get.
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Problems in positive characteristic

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- Description of invariants is harder to get.
- Representation theory is more complicated.
- Commutative algebra statements do not always go through.
Positive characteristic

Example: Matrix invariants

- The traces description misses some invariants.
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\[
\text{char}(A) = \det(I + tA) = \sum_{i=0}^{n} \sigma_i(A)t^i
\]

\[
\sigma_1(A) = \text{Tr}(A)
\]

\[
\sigma_2(A) = \frac{1}{2} (\text{Tr}(A)^2 - \text{Tr}(A^2))
\]

\[
\sigma_n(A) = \det(A).
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- \(\sigma_i\) is also an invariant polynomial.
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- \( \sigma_i \) is also an invariant polynomial.
- In char \( p \), can’t always write \( \sigma_i \) in terms of traces of powers.
- **Donkin:** The set \( \sigma_i(A_w) \) is a generating set of invariants (highly non-trivial!)
Problem 1: Description of invariants

- How does one get a description of invariants?
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- Well understood representation theory $\implies$ multilinear invariants easy to understand.
- Weyl’s method doesn’t work in positive characteristic.
- Weyl’s method gives traces description for matrix invariants.
Example: A strange phenomenon for the cyclic group in \( \text{char } p \)

- **Maschke's theorem**: In char 0, any representation of a finite group splits into a direct sum of irreducibles.
Example: A strange phenomenon for the cyclic group in char $p$

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- But cannot find a complementary subrepresentation.
Problem 2: Reductive ≠ linearly reductive

- In characteristic 0, reductive $\implies$ every representation splits into direct sum of irreducibles (linearly reductive).
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  2. Kempf: Hilbert series is a proper rational function.

- These are not always true for actions of reductive groups.
A general bound on null cone in char 0 exists (due to Popov/Derksen), but this doesn’t hold in char $p$. 
Problem 2 continued..

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- Recall Derksen’s result:

  Degree bounds for generators $= \text{Poly}(\text{degree bounds for null cone})$
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- Curiously, the bound on null cone for matrix semi-invariants holds in all characteristics!
Matrix semi-invariants

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- The theory of good filtrations allows us to pass from results in char 0 to char $p$!
- Have to thank Akin-Buchsbaum-Weyman (80’s), Donkin (90’s) and Hashimoto (00’s) for this wonderful and delicate theory.
Results in positive characteristic (Derksen, Makam)

- **Matrix semi-invariants:** A degree bound for generating invariants of $mn^4$ (and a bound of $n^6$ if $p > \Omega(n^6)$).
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Matrix invariants: A degree bound of $(m + 1)n^4$ for generating invariants (previous known was $O(n^7 m^n)$ due to Domokos).
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- Our algorithm for orbit closure adapts well, and continues to run in polynomial time!
Further questions

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