Colouring graphs with no odd holes

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Chromatic number $\chi(G)$: minimum number of colours needed to colour $G$. 
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Clique number $\omega(G)$: size of largest clique in $G$. 
Theorem (Tutte, 1948)

There are graphs $G$ with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large.
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**Hole:** induced subgraph of $G$ which is a cycle of length $> 3$.

**Antihole:** induced subgraph of $G$ which is the complement of a cycle of length $> 3$. 
Theorem (Tutte, 1948)

*There are graphs G with \( \omega(G) = 2 \) and \( \chi(G) \) arbitrarily large.*

**Hole:** induced subgraph of \( G \) which is a cycle of length \( > 3 \).

**Anti-hole:** induced subgraph of \( G \) which is the complement of a cycle of length \( > 3 \).

Theorem (Chudnovsky, Robertson, S., Thomas, 2006)

*If G has no odd holes and no odd antiholes then \( \chi(G) = \omega(G) \).*
Theorem (Tutte, 1948)

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If $G$ has no odd holes and no odd antiholes then $\chi(G) = \omega(G)$.

What happens in between?
Theorem (Tutte, 1948)

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What happens in between?

Conjecture (Gyárfás, 1985)

If $G$ has no odd holes then $\chi(G)$ is bounded by a function of $\omega(G)$.
Theorem (trivial)

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Theorem (Chudnovsky, Robertson, S., Thomas, 2010)

If $G$ has no odd holes and $\omega(G) = 3$ then $\chi(G) \leq 4$. 
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Theorem (Chudnovsky, Robertson, S., Thomas, 2010)

If $G$ has no odd holes and $\omega(G) = 3$ then $\chi(G) \leq 4$.

Theorem (Scott, S., August 2014)

If $G$ has no odd holes then $\chi(G) \leq 2^{3\omega(G)}$. 
Cograph: graph not containing a 4-vertex path as an induced subgraph.
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Lemma

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**Lemma**
If $J$ is a cograph with $|V(J)| > 1$, then either $J$ or its complement is disconnected.

**Theorem**
Let $G$ be a graph, and let $A, B \subseteq V(G)$ be disjoint, where $A$ is stable and $B \neq \emptyset$. Suppose that

- every vertex in $B$ has a neighbour in $A$;
- there is a cograph $J$ with vertex set $A$, with the property that for every induced path $P$ with ends in $A$ and interior in $B$, its ends are adjacent in $J$ if and only if $P$ has odd length.

Then there is a partition $X, Y$ of $B$ such that every $\omega(G)$-clique in $B$ intersects both $X$ and $Y$. 
The proof

Let $G$ be a graph with no odd hole. We need to show $\chi(G) \leq 2^{3\omega(G)}$. 
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Let $G$ be a graph with no odd hole. We need to show $\chi(G) \leq 2^{3\omega(G)}$.

Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole.

Then $\chi(G) \leq 48n^3$. 
Levelling in $G$: Sequence $L_0, L_1, L_2, \ldots, L_k$ of disjoint subsets of $V(G)$ where

- $|L_0| = 1$
- each vertex in $L_{i+1}$ has a neighbour in $L_i$
- for $j > i + 1$ there are no edges between $L_i$ and $L_j$. 
Levelling in $G$: Sequence $L_0, L_1, L_2, \ldots, L_k$ of disjoint subsets of $V(G)$ where
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Enough to show:

Assume
- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$.

Then $\chi(L_k) \leq 24n^3$. 
Parent of $v \in L_{i+1}$ is a vertex in $L_i$ adjacent to $v$.

$L_i$ has the **single parent property** if $i < k$ and every vertex in $L_i$ is the unique parent of some vertex.
Parent of $v \in L_{i+1}$ is a vertex in $L_i$ adjacent to $v$.

$L_i$ has the **single parent property** if $i < k$ and every vertex in $L_i$ is the unique parent of some vertex.

$L_i$ has the **parity property** if for all $u, v \in L_i$, all induced paths between them with interior in lower levels have the same parity.
Parent of $v \in L_{i+1}$ is a vertex in $L_i$ adjacent to $v$.

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Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-1}$ have the single parent property.

Then $\chi(L_k) \leq 24n^3$. 

Spine: Path $S = s_0-s_1-\cdots-s_k$ where

- $s_i \in L_i$ for all $i$
- $s_i$ is the single parent of $s_{i+1}$ for all $i < k$
- every vertex in $N(S)$ has the same type, and not type 5 or type 6.
Spine: Path $S = s_0-s_1-\cdots-s_k$ where

- $s_i \in L_i$ for all $i$ 
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$N(S)$ is the set of vertices not in $S$ with a neighbour in $S$. 
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- every vertex in $N(S)$ has the same type, and not type 5 or type 6.

$N(S)$ is the set of vertices not in $S$ with a neighbour in $S$.

Type of $v \in N(S) \cap L_i$:
- **Type 1:** $i$ even, $v$ adjacent to $s_{i-1}$ and to no other vertex in $S$
- **Type 2:** $i$ odd, $v$ adjacent to $s_{i-1}$ and to no other vertex in $S$
- **Type 3:** $i$ even, $v$ adjacent to $s_{i-1}, s_i$ and to no other vertex in $S$
- **Type 4:** $i$ odd, $v$ adjacent to $s_{i-1}, s_i$ and to no other vertex in $S$
- **Type 5:** $i$ even, $v$ adjacent to $s_i$ and to no other vertex in $S$
- **Type 6:** $i$ odd, $v$ adjacent to $s_i$ and to no other vertex in $S$. 
Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-1}$ have the single parent property
- there is a spine.

Then $\chi(L_k) \leq 4n^3$. 
$L_i$ satisfies the **parent rule** if all adjacent $u, v \in L_i$ have the same parents.
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**Theorem**

**Suppose**

- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-1}$ have the single parent property
- there is a spine.

Then $L_0, \ldots, L_{k-2}$ satisfy the parent rule.
Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-2}$ satisfy the parent rule.

Then $\chi(L_k) \leq 4n^3$. 
Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-2}$ satisfy the parent rule
- $L_{k-2}$ is stable.

Then $\chi(L_k) \leq 4n^2$. 
Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $G$ has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in $G$
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-2}$ satisfy the parent rule.
- $L_{k-1}$ is stable.

Then $\chi(L_k) \leq 2n$. 
Let $L_0, \ldots, L_t$ be a levelling in $G$, where $L_t$ is stable and has the parity property. The graph of jumps on $L_t$ is the graph on $L_t$, in which $u$, $v$ are adjacent if all induced paths between $u$, $v$ with interior in lower levels are odd.
Let $L_0, \ldots, L_t$ be a levelling in $G$, where $L_t$ is stable and has the parity property.
The graph of jumps on $L_t$ is the graph on $L_t$, in which $u, v$ are adjacent if all induced paths between $u, v$ with interior in lower levels are odd.

**Theorem**

Suppose that
- $G$ has no odd hole
- $L_0, \ldots, L_t$ is a levelling in $G$
- $L_t$ has the parity property
- $L_0, \ldots, L_{t-1}$ satisfy the parent rule
- $L_t$ is stable.

Then the graph of jumps on $L_t$ is a cograph.
Enough to show:

Assume

- Every graph $H$ with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $L_0, \ldots, L_k$ is a levelling in $G$
- $L_{k-1}$ has the parity property
- $L_{k-1}$ is stable
- the graph of jumps on $L_{k-1}$ is a cograph.

Then $\chi(L_k) \leq 2n$. 
Recall:

**Theorem**

Let $G$ be a graph, and let $A, B \subseteq V(G)$ be disjoint, where $A$ is stable and $B \neq \emptyset$. Suppose that

- every vertex in $B$ has a neighbour in $A$;
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Then there is a partition $X, Y$ of $B$ such that every $\omega(G)$-clique in $B$ intersects both $X$ and $Y$. 