On certain topics related to Arthur classification of discrete spectrum

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9/28/2015
Discrete spectrum

- $F$ number field, $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. $G$ quasi-split classical groups. $X_G := Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})$, finite volume.
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$L^2(X_G)$ is the space of square-integrable functions $\phi : X_G \rightarrow \mathbb{C}$. $L^2(X_G)$ is a $G(\mathbb{A})$-module by action: $g \cdot \phi(x) := \phi(xg)$.

$L^2_d(X_G) = \bigoplus$ irreducible $G(\mathbb{A})$-submodules in $L^2(X_G)$. $\phi \in L^2_d(X_G)$ is called cuspidal if all constant terms are zero: 

$$\int_{N(F)} \int_{N(\mathbb{A})} \phi(n) dn \equiv 0,$$

where $N$ is the unipotent radical of any proper parabolic subgroup of $G$.

$L^2_c(X_G) = \{$ cuspidal functions $\} -$ cuspidal spectrum.

$L^2_r(X_G) = L^2_d(X_G) \oplus L^2_r(X_G)$ - residual spectrum, consists of residues of Eisenstein series, by the theory of Eisenstein series of Langlands.
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Classification of discrete spectrum

  \[ L^2_d(X_{GL_n}) = \{ \Delta(\tau, b) | \tau \text{ cuspidal representation of } GL_a(\mathbb{A}), n = ab \}. \]

Quasi-split classical groups: Arthur ($Sp_{2n}$, $SO_n$), Mok ($U_n$).

Theorem (Arthur 2013)

\[ L^2_d(X_{Sp_{2n}}) = \bigoplus_{\psi \in \tilde{\Psi}_2(Sp_{2n})} \bigoplus_{\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)} \pi. \]

$\tilde{\Psi}_2(Sp_{2n})$ the set of (global) Arthur parameters:

- $\psi_i = (\tau_i, b_i)$, pairwise different, called simple Arthur parameters,
  $\tau_i$ cuspidal representation of $GL_{a_i}(\mathbb{A})$, $b_i \in \mathbb{Z} \geq 1$, $\sum b_i = 2n + 1$.

Each $\psi_i$ is of orthogonal type ($\hat{Sp}_{2n} = SO_{2n+1}(C) \hookrightarrow \to GL_{2n+1}(C)$):

- $b_i$ even, $\tau_i$ of symplectic type, i.e., $L(s, \tau_i)$ has a pole at $s = 1$;
- $b_i$ odd, $\tau_i$ of orthogonal type, i.e., $L(s, \tau_i, \Sym^2)$ has a pole at $s = 1$.

$\tilde{\Pi}_\psi(\epsilon_\psi)$ is endoscopic lifting to the isobaric sum representation $\Delta(\tau_1, b_1) \boxplus \cdots \boxplus \Delta(\tau_r, b_r)$ of $GL_{2n+1}(\mathbb{A})$. 
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Fourier Coefficients of Automorphic Forms

- $GL_n$, $U_n$ standard maximal unipotent subgroup, not abelian if $n \geq 3$. Idea of Shalika (1974) and Piatetski-Shapiro (1979): taking Fourier expansion column-by-column, prove that cuspidal automorphic representations of $GL_n(\mathbb{A})$ are generic, i.e., having nonzero Whittaker Fourier coefficients.

In general, one can define Fourier coefficients from nilpotent orbits. In this way, one can measure the size of Fourier coefficients: $n(\pi) = \{\text{all nilpotent orbits providing nonzero FC for } \pi\} \supset n_m(\pi)$.

For quasi-split classical groups $G$, nilpotent orbits $\leftrightarrow \{\text{partitions, quadratic forms}\}$.

$p(\pi) = \{\text{all partitions parametrizing orbits in } n(\pi)\} \supset p_m(\pi)$. 
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Fourier coefficients of $\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$

- **Shahidi Conjecture**: for generic Arthur parameter $\psi = \bigoplus_i (\tau_i, 1)$, $\exists \pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$, generic, i.e., has nonzero Whittaker Fourier coefficients $\leftrightarrow$ regular orbits $\leftrightarrow$ maximal partition.

  Proved for quasi-split classical groups, by Ginzburg-Rallis-Soudry, using the theory of automorphic descent.
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- Jiang Conjecture (generalization of Shahidi conjecture): for any $\psi$, there is a partition $p_\psi$ (depends on $\psi$), such that

  1. $p_\psi$ is an upperbound (dominant order) for $p^m(\pi)$, $\forall \pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$;
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- **Known result for $Sp_{2n}$ (Jiang-L)**:
  1. $p_\psi$ is an upperbound (dictionary order) for $p^m(\pi)$, $\forall \pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$;
  2. for $\psi = \boxplus_i (\tau_i, 1) \boxplus (\tau, b)$, $b > 1$, there exists $\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$, such that $p_\psi \in p^m(\pi)$. 
Explicit construction of $\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$

- Construction of residual representations, $G$ quasi-split classical groups:
  2. Jiang-L-Zhang (2013): calculate all possible poles on the right half plane of the Eisenstein series define from $\text{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})} \Delta(\tau, b)|s \otimes \sigma$, obtain residual representations in certain $\tilde{\Pi}_\psi(\varepsilon_\psi)$.
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- Construction of cuspidal representations:

- Jiang (2014) proposes a general framework which may give explicit construction of $\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$. 
Topics related to Arthur classification of discrete spectrum

Relations between $\tilde{\Pi}_\psi(\varepsilon_\psi)$ of different groups

- Automorphic descent:
  representations of $Sp_{2n}(\mathbb{A}) \to$ representations of $\tilde{Sp}_{2k}(\mathbb{A})$,
  representations of $SO_{2n+1}(\mathbb{A}) \to$ representations of $SO_{2k}(\mathbb{A})$, etc.
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- Ginzburg-Jiang-Soudry (2012):

\[
\begin{align*}
& Sp_{4n}(\mathbb{A}) & \mathcal{N}_{4n} \quad \dot{\cup} \quad \mathcal{N}'_{4n} & (\tau, 2) \boxplus (1, 1) \\
& \downarrow \quad \mathcal{D}_{2n, \psi^{-1}}^{4n} & \\
& \tilde{Sp}_{2n}(\mathbb{A}) & \tilde{\mathcal{N}}_{2n} \quad \dot{\cup} \quad \tilde{\mathcal{N}}'_{2n} & (\tau, 1)
\end{align*}
\]

$\tau$ cuspidal representation of $GL_{2n}(\mathbb{A})$ of symplectic type, $L\left(\frac{1}{2}, \tau\right) \neq 0$. 
Relations between $\tilde{\Pi}_\psi(\varepsilon_\psi)$ of different groups, continue

- L (2013): extend to higher rank cases:

$$\downarrow \quad \downarrow$$

$$\tilde{Sp}_{4mn+2n}(A) \quad \tilde{N}_{4mn+2n} \cup \tilde{N}'_{4mn+2n}$$

$$\downarrow \quad \downarrow$$

$$Sp_{4mn}(A) \quad N_{4mn} \cup N'_{4mn}$$

$$\downarrow \quad \downarrow$$

$$\tilde{Sp}_{4mn-2n}(A) \quad \tilde{N}_{4mn-2n} \cup \tilde{N}'_{4mn-2n}$$

$$\downarrow \quad \downarrow$$

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$$\mathcal{D}_{2n,\psi}^{4mn} \quad \mathcal{D}_{2n,\psi}^{4mn+2n} \quad \mathcal{D}_{2n,\psi}^{4mn+2n}$$
Non-cuspidality of $\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)$

1. Piatetski-Shapiro (1983), Soudry (1988), there exits $\psi$ for $GSp_4$ such that there are no cuspidal representations in $\tilde{\Pi}_\psi(\varepsilon_\psi)$.

2. Paniagua-Taboada (2011), split $SO_{4n}$, $\psi = (\tau, 2n)$, $\tau$ cuspidal representation of $GL_2(\mathbb{A})$ of symplectic type, there are no cuspidal representations in $\tilde{\Pi}_\psi(\varepsilon_\psi)$ (can be reproved using FC of automorphic forms).
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Examples: $F = \mathbb{Q}(i)$, $\psi = (1, b_1) \boxtimes (\tau, b_2)$ for $Sp_{b_1+2b_2-1}$, $\tau$ cuspidal representation of $GL_2(\mathbb{A})$ of symplectic type, $b_1$ odd, $b_2$ even. Then $\tilde{\Pi}_\psi(\varepsilon_\psi)$ may contain cuspidal representation only when

$$(b_1, b_2) = (1, 2), (3, 2), (5, 2), \text{ or } (1, 4).$$
Thank you!