On certain topics related to Arthur classification of discrete spectrum

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Discrete spectrum

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- $L^2(X_G)$ is the space of square-integrable functions $\phi : X_G \to \mathbb{C}$. $L^2(X_G)$ is a $G(\mathbb{A})$-module by action: $g \cdot \phi(x) := \phi(xg)$. $L^2_d(X_G) = \oplus$ irreducible $G(\mathbb{A})$-submodules in $L^2(X_G)$.

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- $\phi \in L^2_d(X_G)$ is called cuspidal if all constant terms are zero:

$$\int_{N(F)\backslash N(\mathbb{A})} \phi(n)dn \equiv 0,$$

where N is the unipotent radical of any proper parabolic subgroup of G. $L^2_c(X_G) = \{\text{cuspidal functions}\}$ - cuspidal spectrum.

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- $L^2_d(X_G) = L^2_c(X_G) \oplus L^2_r(X_G)$. $L^2_r(X_G)$ - residual spectrum, consists of residues of Eisenstein series, by the theory of Eisenstein series of Langlands.

Classification of discrete spectrum

- GL_n: Jacquet (1984), Moeglin-Waldspurger (1989).
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Theorem (Arthur 2013)

$L^2_d(X_{Sp_{2n}}) = \oplus_{\psi \in \widetilde{\Psi}_2(Sp_{2n})} \oplus_{\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)} \pi$.

$\widetilde{\Psi}_2(Sp_{2n})$ the set of (global) Arthur parameters: $\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$.
$\psi_i = (\tau_i, b_i)$, pairwise different, called simple Arthur parameters, τ_i cuspidal representation of $GL_{a_i}(\mathbb{A})$, $b_i \in \mathbb{Z}_{\geq 1}$, $\sum_{i=1}^r a_i b_i = 2n + 1$.

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- Each ψ_i is of orthogonal type ($\widehat{Sp_{2n}} = SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$):
  1. b_i even, τ_i of symplectic type, i.e., $L(s, \tau_i, \wedge^2)$ has a pole at $s = 1$;
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- $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ is endoscopic lifting to the isobaric sum representation $\Delta(\tau_1, b_1) \boxplus \cdots \boxplus \Delta(\tau_r, b_r)$ of $GL_{2n+1}(\mathbb{A})$.

Fourier Coefficients of Automorphic Forms

- GL_n, U_n standard maximal unipotent subgroup, not abelian if $n \geq 3$. Idea of Shalika (1974) and Piatetski-Shapiro (1979): taking Fourier expansion column-by-column, prove that cuspidal automorphic representations of $GL_n(\mathbb{A})$ are generic, i.e., having nonzero Whittaker Fourier coefficients.

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- For quasi-split classical groups G,

$$\text{nilpotent orbits} \leftrightarrow \{(\text{partitions}, \text{quadratic forms})\}.$$

$$\mathfrak{p}(\pi) = \{\text{all partitions parametrizing orbits in } \mathfrak{n}(\pi)\} \supset \mathfrak{p}^m(\pi).$$

Fourier coefficients of $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$

- Shahidi Conjecture: for generic Arthur parameter $\psi = \boxplus_i (\tau_i, 1)$, $\exists \pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, generic, i.e., has nonzero Whittaker Fourier coefficients $\leftrightarrow$ regular orbits $\leftrightarrow$ maximal partition.
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- Jiang Conjecture (generalization of Shahidi conjecture): for any ψ, there is a partition $\underline{p}_\psi$ (depends on ψ), such that
  1. $\underline{p}_\psi$ is an upperbound (dominant order) for $\mathfrak{p}^m(\pi)$, $\forall \pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$;
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- Known result for Sp_{2n} (Jiang-L):
  1. $\underline{p}_\psi$ is an upperbound (dictionary order) for $\mathfrak{p}^m(\pi)$, $\forall \pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$;
  2. for $\psi = \boxplus_i(\tau_i, 1) \boxplus (\tau, b)$, $b > 1$, there exists $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$, such that $\underline{p}_\psi \in \mathfrak{p}^m(\pi)$.

Explicit construction of $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$

- Construction of residual representations, G quasi-split classical groups:
  1. Moeglin (2008, 2011), makes local/global conjectures towards existence of residual representations in $\widetilde{\Pi}_\psi(\varepsilon_\psi)$.
  2. Jiang-L-Zhang (2013): calculate all possible poles on the right half plane of the Eisenstein series define from $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\Delta(\tau, b)|\cdot|^s \otimes \sigma$, obtain residual representations in certain $\widetilde{\Pi}_\psi(\varepsilon_\psi)$.

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- Construction of cuspidal representations:
  1. Piatetski-Shapiro (1983), Soudry (1988), construct all cuspidal representations in non-generic packets of $GSp_4(\mathbb{A})$, using the theory of theta correspondence.
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- Jiang (2014) proposes a general framework which may give explicit construction of $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$.

Relations between $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ of different groups

- Automorphic descent:
  representations of $Sp_{2n}(\mathbb{A}) \to$ representations of $\widetilde{Sp}_{2k}(\mathbb{A})$,
  representations of $SO_{2n+1}(\mathbb{A}) \to$ representations of $SO_{2k}(\mathbb{A})$, etc.

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- Ginzburg-Jiang-Soudry (2012):

$$
\begin{array}{lccccc}
\mathrm{Sp}_{4n}(\mathbb{A}) & & \mathcal{N}_{4n} & \dot{\bigcup} & \mathcal{N}'_{4n} & (\tau,2)\boxplus(1,1) \\
 & & & \downarrow \mathcal{D}^{4n}_{2n,\psi^{-1}} & & \\
\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A}) & \widetilde{\mathcal{N}}_{2n} \quad \dot{\bigcup} & \widetilde{\mathcal{N}}'_{2n} & & & (\tau,1)
\end{array}
$$

τ cuspidal representation of $GL_{2n}(\mathbb{A})$ of symplectic type, $L(\frac{1}{2},\tau) \neq 0$.

Relations between $\widetilde{\Pi}_{\psi}(\varepsilon_{\psi})$ of different groups, continue

- L (2013): extend to higher rank cases:

$$
\begin{array}{llll}
\vdots & & & \vdots \\
 & & & \downarrow \mathcal{D}^{4mn}_{2n,\psi} \\
\widetilde{\mathrm{Sp}}_{4mn+2n}(\mathbb{A}) & & & \widetilde{\mathcal{N}}_{4mn+2n} \ \dot{\bigcup} \ \widetilde{\mathcal{N}}'_{4mn+2n} \\
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\mathrm{Sp}_{4mn}(\mathbb{A}) & & \mathcal{N}_{4mn} \ \dot{\bigcup} \ \mathcal{N}'_{4mn} & \\
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\widetilde{\mathrm{Sp}}_{4mn-2n}(\mathbb{A}) & \widetilde{\mathcal{N}}_{4mn-2n} \ \dot{\bigcup} \ \widetilde{\mathcal{N}}'_{4mn-2n} & & \\
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Non-cuspidality of $\pi \in \widetilde{\Pi}_\psi(\varepsilon_\psi)$

- 1. Piatetski-Shapiro (1983), Soudry (1988), there exits ψ for GSp_4 such that there are no cuspidal representations in $\widetilde{\Pi}_\psi(\varepsilon_\psi)$.
  2. Paniagua-Taboada (2011), split SO_{4n}, $\psi = (\tau, 2n)$, τ cuspidal representation of $GL_2(\mathbb{A})$ of symplectic type, there are no cuspidal representations in $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ (can be reproved using FC of automorphic forms).

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- Examples: $F = \mathbb{Q}(i)$, $\psi = (1, b_1) \boxplus (\tau, b_2)$ for $Sp_{b_1+2b_2-1}$, τ cuspidal representation of $GL_2(\mathbb{A})$ of symplectic type, b_1 odd, b_2 even. Then $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ may contain cuspidal representation only when

$$(b_1, b_2) = (1, 2), (3, 2), (5, 2), \text{ or,} (1, 4).$$

- Thank you!