A SPECTRAL GAP THEOREM IN $SL_2(\mathbb{R})$ AND APPLICATIONS

BACKGROUND: EXPANSION IN UNITARY GROUPS

$g_1, \ldots, g_k \in SU(2)$ algebraic and free

$T : L^2(G) \to L^2(G)$ Hecke operator

$$Tf(x) = \sum \left(f(g_j x) + f(g_j^{-1} x)\right)$$

THEOREM [B-G] *There is spectral gap*

$$\lambda_1(T) < 2k - \gamma$$

$\gamma = \gamma(g_1, \ldots, g_k)$ controlled by non commutative diophantine property

Applications to **tilings** (**Conway-Radin**) and **quantum computation** (**Solovay-Kitaev)**

NON COMMUTATIVE DIOPHANTINE PROPERTY

$$\mathcal{G} = \{g_1, \dots, g_k\}$$

$$W_\ell(\mathcal{G}) = \text{ words of length } \ell$$

DC: $g \in W_\ell(\mathcal{G}) \backslash \{1\} \Rightarrow \|1 - g\| > A^{-\ell}$

Satisfied for $\mathcal{G} \subset$ Mat$_{2\times 2}(\overline{\mathbb{Q}})$ where A depends on the height

(**Gamburd-Jakobson-Sarnak**)

$SU(d)$

$$g_1, \dots, g_k \in SU(d) \cap \text{Mat}_{d\times d}(\overline{\mathbb{Q}})$$

$$\Gamma = \langle g_1, \dots, g_k \rangle$$

Assume Γ topologically dense

$$Tf(x) = \sum \big(f(g_j x) + f(g_j^{-1} x)\big)$$

THEOREM [**B-G**]

T has spectral gap

Generalized to compact simple Lie groups

(**Benoist–De Saxe**)

A SPECTRAL GAP PROPERTY FOR THE PROJECTIVE ACTION OF $SL_2(\mathbb{R})$

$$P_1\ (\mathbb{R}) \simeq \mathbb{T} = \mathbb{R} / \mathbb{Z}$$

ρ = projective representation of $SL_2(\mathbb{R})$ on $L^2(\mathbb{T})$

$$\rho_g f = \left(\tau'_{g^{-1}}\right)^{\frac{1}{2}} \left(f \circ \tau_{g^{-1}}\right)$$

τ_g = projective action of g

THEOREM [B-Y] *Given* $0 < c < 1$, *there is* $k_0 \in \mathbb{Z}_+$ *such that the following holds.*
Let $\mathcal{G} \subset SL_2(\mathbb{R})$, $|\mathcal{G}| = k > k_0$, *generating freely the free group on* k *generators. Assume moreover*

$\|g - 1\| < 1/k$ *for* $g \in \mathcal{G}$

$\|g - 1\| > k^{-\ell/c}$ *for* $g \in W_\ell(\mathcal{G})\backslash\{1\}$ *and* ℓ *arbitrary*

Then

$$\Big\|\frac{1}{2k}\sum_{g\in\mathcal{G}}(\rho_g f + \rho_{g^{-1}} f)\Big\|_2 \leq \frac{1}{2}\|f\|_2$$

provided $f \perp V$, *where*

$$V = [e(n\theta); |n| < K] \subset L^2(\mathbb{T}), K = K(k)$$

MOTIVATIONS

- Theoretical computer science

- Absolute continuity of Furstenberg measures

- The Anderson-Bernoulli model in Physics

DEFINITION *A monotone expander is a finite family* Ψ *of maps* ψ *from a sub-interval of* $[0,1]$ *to* $[0,1]$ *such that*

- *There is a constant* $c > 0$ *such that for any* $A \subset [0,1],\ |A| \leq \frac{1}{2}$

$$|\Psi(A)| \geq (1+c)|A| \qquad \Psi(A) = \bigcup_{\psi \in \Psi} \psi(A)$$

- *Every* $\psi \in \Psi$ *is continuous and monotone*

THEOREM [B-Y] *There exists (explicit) monotone expanders*

MAIN INGREDIENT: Projective action of family $\mathcal{G}$ satisfying previous theorem

DIMENSIONAL EXPANDERS

DEFINITION $\mathbb{F}$ = *field. A dimension expander over* $\mathbb{F}^n$ *is a constant number of matrices* $M_1, \ldots M_k$ *in* $\mathbb{F}^{n \times n}$ *for which there is a constant* $c > 0$ *(*c, k *independent of* n*) such that*

$$\dim[M_1(V) \bigcup \cdots \bigcup M_k(V)] > (1 + c) \dim V$$

for any subspace V *of* $\mathbb{F}^n$, $\dim V \leq \frac{n}{2}$

For char $\mathbb{F} = 0$, existence proven by **Lubotzky-Zelmanov** using property τ

Dvir-Shpilka, Dvir-Wigderson monotone expanders $\Rightarrow$ dimensional expanders

COROLLARY *Existence of dimension expanders for arbitrary fields*

PUSHDOWN GRAPHS AND TURING MACHINES

DEFINITION (**Pippenger, Paul-Pippenger-Szemeredi-Trotter**)

A d-pushdown graph is a graph on an ordered set of vertices such that when ordered along the spine of a book, the edges can be drawn on d pages and in each page the edges do not touch

Hopcroft-Paul-Valiant Relation to complexity and Turing machines

Dvir-Wigderson Relation to monotone expanders

COROLLARY 4 *There exists (explicit) d-pushdown expanders*

$\Rightarrow$ *no sub-linear size separators*

$\longleftrightarrow$

THEOREM (**Lipton-Tarjan**) *Planar graphs have $0(\sqrt{n})$-size separators*

FURSTENBERG MEASURES

$\nu =$ probability measure on $SL_2(\mathbb{R})$ with proximal and strongly irreducible action.

Furstenberg measure μ is a unique ν-stationary measure on $P_1(\mathbb{R})$, i.e.

$$\int f d\mu = \sum_g \nu(g) \int (f \circ \tau_g) d\mu$$

PROBLEM (**Kaimanovich-Le Prince**)
Can the Furstenberg measure of a finitely supported (symmetric) probability measure on $SL_2(\mathbb{R})$ be absolutely continuous?

ANSWER YES

THEOREM *There are examples with* $\frac{d\mu}{d\sigma}$ *arbitrarily smooth*

Barany-Pollicott-Simon: Examples of ac-stationary measures for **non-symmetric** random walks

SPECTRAL THEORY

OF LATTICE SCHRÖDINGER OPERATORS

$$H = \Delta + \lambda V \text{ on } \mathbb{Z}$$

$$\Delta(i,j) = \begin{cases} 1 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise} \end{cases} \quad \text{(lattice Laplacian)}$$

$V = \sum V_i \delta_i$ with V_i chosen independently according to distribution μ

λ = coupling

Introduced by **Anderson** to model transport in inhomogenous media

THEOREM (**Frohlich-Spencer**)

In $1D$*, at any disorder* $\lambda \neq 0$*, for almost all realization of* V*,* H *has pure point spectrum with exponentially decaying eigenfunctions*

(**Anderson Localization**)

CONJECTURES

2D AL

3D An ac-component in the bulk of the spectrum

(AL persists for large λ and at edge of the spectrum)

THEOREM (**Simon-Taylor**, 85)

Assume V *distributed according to measure* μ *on* $\mathbb{R}$*,* $\mu \ll$ *Lebesgue and* $\frac{d\mu}{dx} \in L^1_\alpha$ *for some* $\alpha > 0$

Then the integrated density of states k *of* H *is* C^∞

THEOREM (**Germinet-Klopp**, 2011)

Same assumption on V*. Then local eigenvalue statistics of* H *are Poisson*

THEOREM (**B**, 2011)

Same conclusions hold if $\dim \mu > 0$ *(Holder potentials)*

PROBLEM What happens in the Bernoulli case for small λ?

DENSITY OF STATES
OF THE ANDERSON-BERNOULLI MODEL

$\mathcal{N}$ = Integrated density of sates (IDS) $\frac{d\mathcal{N}}{dE} = k$

THEOREM $\mathcal{N}$ *is Hölder regular*

- **Carmona-Klein-Martinelli** (87) Le Page's method
- **Shubin-Vakilian-Wolff** (88) Several proofs using harmonic analysis and the uncertainty principle

HALPERIN: $\mathcal{N}$ *is not Hölder continuous of any order* $\alpha > \frac{2\log 2}{\mathit{Arccosh}(1+\lambda)}$

CONJECTURE *For* λ *sufficiently small,* k *is bounded and becomes arbitrary smooth for* $\lambda \to 0$

THEOREM [**B**, 012] $\alpha(\lambda) \to 1$ *for* $\lambda \to 0$

THEOREM [**B**, 013] *Let H_λ be the Anderson-Bernoulli Hamiltonian with coupling λ and restrict the energy $|E| < 2-\delta$ for some fixed $\delta > 0$.*

Given a constant $C > 0$ and $s \in \mathbb{Z}_+$, there is $\lambda_0 = \lambda_0(C, s)$ such that $\mathcal{N}(E)$ is C^s-smooth provided λ satisfies the following conditions

- $|\lambda| < \lambda_0$

- *λ is an algebraic number of degree $d < C$ and minimal polynomial $P_d(x) \in \mathbb{Z}[X]$ with coefficients bounded by $\left(\frac{1}{\lambda}\right)^C$*

- *λ has a conjugate λ' of modulus $|\lambda'| \geq 1$*

RELATION TO $SL_2(\mathbb{R})$: SCHRÖDINGER COCYCLES

THE TRANSFER MATRIX FORMALISM

Equation $H\xi = E\xi$ equivalent to $\begin{pmatrix} \xi_{n+1} \\ \xi_n \end{pmatrix} = M_N(E) \begin{pmatrix} \xi_1 \\ \xi_0 \end{pmatrix}$

$$M_N(E) = \begin{pmatrix} E - \lambda V_N & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - \lambda V_1 & -1 \\ 1 & 0 \end{pmatrix}$$

LYAPOUNOV EXPONENT $\quad L(E) = \lim_{N\to\infty} \frac{1}{N} \log \|M_N(E)\|$

THOULESS FORMULA $\quad L(E) = \int \log |E - E'| d\mathcal{N}(E')$

Role of **Furstenberg**'s theory of random matrix products

A NEW SPECTRAL GAP

Take λ as above and E arbitrary. Set

$$g_+ = \begin{pmatrix} E+\lambda & -1 \\ 1 & 0 \end{pmatrix} \qquad g_- = \begin{pmatrix} E-\lambda & -1 \\ 1 & 0 \end{pmatrix}$$

$$h_1 = g_+ g_-^{-1} = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix} \qquad h_2 = g_+^{-1} g_- = \begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix}$$

$V = [e(n\theta); |n| < K]$ with $K = K(\lambda)$ sufficiently large

PROPOSITION $\|f - \rho_{h_1} f\|_2 + \|f - \rho_{h_2} f\|_2 \geq \lambda^\tau \|f\|_2$ *for* $f \in V^\perp$ *and where* τ *can be made arbitrarily small when* $\lambda \to 0$

COROLLARY $\|f - \rho_{g_+} f\|_2 + \|f - \rho_{g_-} f\|_2 \geq \frac{1}{2}\lambda^\tau \|f\|_2$ *for* $f \in V^\perp$

$\leftrightarrow$ $\|f - \rho_{g_+} f\|_2 + \|f - \rho_{g_-} f\|_2 \geq c\lambda \|f\|_2$ *for all* $f \in L^2(\mathbb{T})$

SKETCH OF THE PROOF

LEMMA (**Sanov, Brenner**) *If $|\mu| \geq 2$, then the group generated by*

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

is free

Since λ has conjugate $\lambda', |\lambda'| > 1$, it follows that h_1, h_2 generate a free group

Set $k = \lambda^{-\tau}$ and $\mathcal{G} = \{h_1^\ell h_2^\ell; 1 \leq \ell \leq k\} \subset SL_2(\mathbb{R})$

Then $\mathcal{G}$ are free generators of free group F_k and satisfies conditions of the expansion theorem (DC follows from height considerations going back to [G-J-S])

Hence for $f \in V^\perp, \|f\|_2 = 1$

$$\max_{g \in \mathcal{G}} \|f - \rho_g f\|_2 > \frac{1}{2} \Rightarrow \|f - \rho_{h_1} f\|_2 + \|f - \rho_{h_2} f\|_2 > \frac{1}{2k}$$

FREE GROUPS GENERATED BY PAIRS OF PARABOLIC ELEMENTS

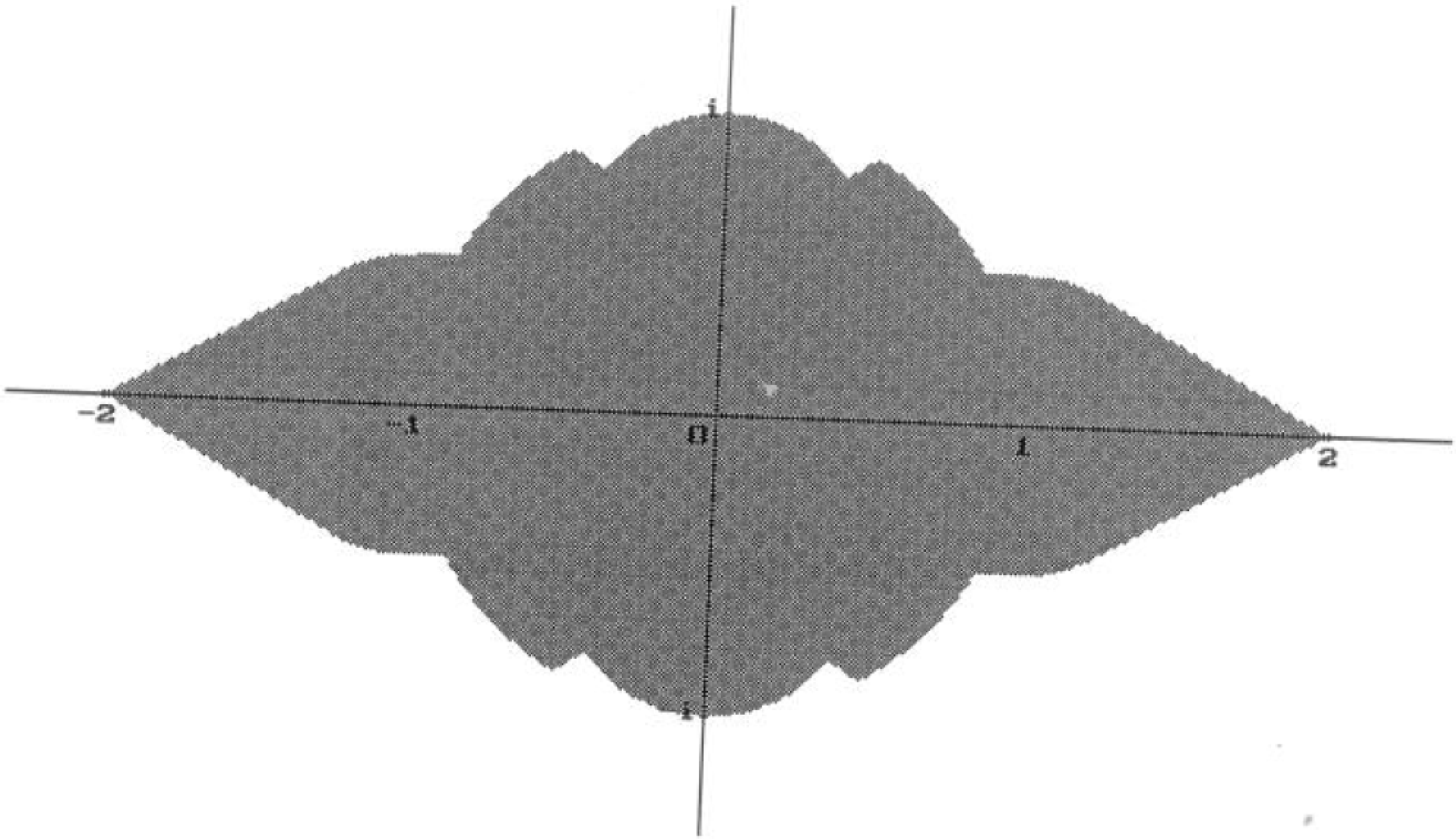
$$\left\langle \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \right\rangle \leftrightarrow G_\lambda = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \right\rangle \text{ for } \mu^2 = 2\lambda$$

DEFINITION $\lambda \in \mathbb{C}$ is called **FREE** if G_λ is a free group

THEOREM λ *is free in the following cases*

- $|\lambda| \geq 2$ (**Brenner**, 55; **Sanov**, 47)
- $|\lambda| \geq 1, |\lambda \pm 1| \geq 1$ (**Chang-Jennings-Ree**, 58)
- $\lambda \pm \frac{i}{2}| \geq \frac{1}{2}, |\lambda \pm 1| \geq 1$ (**Lyubich-Suvorov**, 69)
- $\lambda \notin$ convhull $(|z| = 1, \pm 2)$ (——)
- $|\lambda - 1| > \frac{1}{2}$ and $1 \leq |\text{Re}\,\lambda| < \frac{5}{4}$ (**Ignatov**, 76)
- $|\lambda| > 1$ and $|\text{Im}\,\lambda| \geq \frac{1}{2}$ (——, 79)

Algebraic free points are dense in $\mathbb{C}$ (C-J-R, 58)



Known free points in the complex plane (unshaded)

AVERAGING OPERATORS AND SMOOTHING ESTIMATES

Set

$$Tf = \frac{1}{3}(f + f \circ \tau_{g_+} + f \circ \tau_{g_-})$$

Using the spectral gap, one proves the following

LEMMA *For $|\lambda| < \lambda(s)$ and λ satisfying the conditions of the spectral gap statement, T has the following smoothing property* $(s > 0)$

$$\|T^m f\|_{H^s} \leq C\|f\|_2 + Ce^{-cm}\|f\|_{H^s}$$

$$\|T^m f\|_2 \leq C\|f\|_{H^{-s}} + Ce^{-cm}\|f\|_2$$

COROLLARY *Furstenberg measures μ_E are a.c. with smooth density*

USE OF LARGE DEVIATION ESTIMATES

PROPOSITION *Let*

$$\nu = \frac{1}{2}(\delta_{g_+} + \delta_{g_-})$$

Then

$$\left\| \sum_g (f \circ \tau_g)\nu^{(\ell)}(g) - \int f d\mu \right\|_\infty \leq Ce^{-c\ell}\|f\|_{C^1}$$

COROLLARY

$$\|T^\ell f - \int f d\mu\|_\infty \leq Ce^{-c\ell}\|f\|_{C^1}$$

Together with the LEMMA, this implies

COROLLARY

$$\|(T^\ell f)'\|_{H^s} \leq Ce^{-c\ell}\|f\|_{H^{s+1}}$$

SMOOTHNESS OF LYAPOUNOV EXPONENT AND DENSITY OF STATES

Recall that by Thouless' formula, $L(E)$ and the IDS $\mathcal{N}(E)$ are related by the Hilbert transform

Also

$$L(E) = \int \underset{\pm}{Av} \log \left\| \begin{pmatrix} E \pm \lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\| \mu_E(d\theta) = \int \Phi_E(\theta)\mu_E(d\theta)$$

Since

$$(T_E)^\ell \Phi_E \to L(E)$$

it will suffice to establish bounds on $\partial_E^{(\alpha)}(T_E^\ell \Phi_E)$ which are uniform in ℓ.

Chain rule and $\partial_E \tau_g = -\sin^2 \tau_g$ implies

$$\partial_E(T_E^\ell \Phi_E) = T^\ell(\partial_E \Phi_E) - \sum_{m=1}^{\ell} T^{\ell-m+1}[(T^{m-1}\Phi_E)' \sin^2\theta]$$

$$|\partial_E(T_E^\ell \Phi_E)| < C + \sum_m \|(T^{m-1}\Phi_E)'\|_\infty < \sum e^{-cm} < C$$

Higher order derivatives estimated similarly

LOCAL EIGENVALUE STATISTICS

Assume H has bounded density of states.

Denote H_N the restriction of H to $[1, N]$ with Dirichlet bc

The following statement improves the **GERMINET-KLOPP** result in $1D$

THEOREM *Assume*

- *Furstenberg measures are absolutely continuous with bounded density*
- *Density of states k is continuous*

Fix $E_0 \in \mathbb{R}$ and $I = \left[E_0, E_0 + \frac{L}{N}\right]$ where we let first $N \to \infty$ then $L \to \infty$. The rescaled eigenvalues $\{N(E - E_0) 1_I(E)\}_{E \in \text{ Spec } H_N}$ obey Poisson statistics

COROLLARY *For suitable λ, the local eigenvalue statistics of the Anderson-Bernoulli Hamiltonian H_λ are Poisson*

WEGNER AND MINAMI ESTIMATES

Assume H satisfies conditions of the Theorem

Let $N \in \mathbb{Z}_+$ be large, $I = [E_0 - \delta, E_0 + \delta]$ with $\log \frac{1}{\delta} < \sqrt{N}$

PROPOSITION (**Wegner type estimate**)

$$\mathbb{E}[Tr 1_I(H_N)] = Nk(E_0)|I| + O\Big(N\delta^2 + \delta \log^2 \Big(N + \frac{1}{\delta}\Big)\Big)$$

PROPOSITION (**Minami type estimate**)

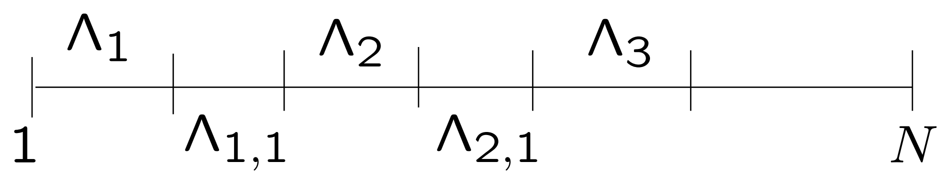
$$\mathbb{E}[H_N \text{ has at least two eigenvalues in } I] \leq$$
$$CN^2\delta^2 + C\delta \log \Big(N + \frac{1}{\delta}\Big)$$

POISSON STATISTICS (SKETCH)

3 ingredients

- Anderson localization
- Wegner estimate
- Minami estimate

$$\Lambda = [1, N] = \Lambda_1 \cup \Lambda_{1,1} \cup \Lambda_2 \cup \Lambda_{2,1} \cup \dots$$



$|\Lambda_\alpha| = M \sim (\log N)^4$

$|\Lambda_{\alpha,1}| = M_1 \sim (\log N)^3$

$\mathcal{E}_\alpha =$ eigenvalues of H_Λ with center of localization in Λ_α

$\mathcal{E}_{\alpha,1} =$ ———————— $\Lambda_{\alpha,1}$

$$\operatorname{Spec} H_\Lambda = \bigcup_\alpha \mathcal{E}_\alpha \cup \bigcup_\alpha \mathcal{E}_{\alpha,1}$$

Λ'_α = neighborhood of Λ_α of size $(\log N)^2$

$\Lambda'_{\alpha,1}$ = —— $\Lambda'_{\alpha,1}$ ——

Anderson localization implies that (with high probability)

dist $(E,$ Spec $H_{\Lambda'_\alpha}) < \frac{1}{N^A}$ for $E \in \mathcal{E}_\alpha$

dist $(E,$ Spec $H_{\Lambda'_{\alpha,1}}) < \frac{1}{N^A}$ for $E \in \mathcal{E}_{\alpha,1}$

Set $I = [E_0, E_0 + \frac{L}{N}]$ fixed interval, $k(E_0) > 0$

Wegner $\Rightarrow \mathbb{E}[|\bigcup_\alpha \mathcal{E}_{\alpha,1} \cap I|] \leq \sum_\alpha \mathbb{E}[Tr 1_{\tilde{I}}(H_{\Lambda'_{\alpha,1}})] < C\frac{N}{M} M_1 \delta < C\frac{L}{\log N} < o(1)$

Minami $\Rightarrow$ $\sum_\alpha \mathbb{E}[H_{\Lambda'_\alpha}$ has two eigenvalues in $I]$

$$< C\frac{N}{M}\Big(M^2\Big(\frac{L}{N}\Big)^2 + \frac{L}{N}\log N\Big) < C\frac{L}{M}\log N < o(1)$$

Introduce (partially defined) random variables

$$E_\alpha = E\,1_I(E) \quad \text{provided} \quad \big|\text{Spec}\, H_{\Lambda'_\alpha} \cap I\big| \le 1$$

Then $\{E_\alpha\}$ take values in I, are independent and have the same distribution

Let $J \subset I$ be an interval, $|J| \sim \frac{1}{N}$

$$\begin{aligned}
\mathbb{E}[1_J(E_\alpha)] &= \mathbb{E}[Tr 1_J(H_{\Lambda'_\alpha})] + O\Big(\frac{\log N}{N}\Big) \\
&= \big(M + O(\log^2 N)\big)\Big(k(E_0) + o(1)\Big)|J| + O\Big(\frac{\log N}{N}\Big) \\
&= Mk(E_0)|J|\Big(1 + O\Big(\frac{1}{\log N}\Big)\Big)
\end{aligned}$$