Quantitative decompositions of Lipschitz mappings

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joint work with Raanan Schul, Stony Brook University

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IAS Analysis Seminar

May 12, 2020
The main broad question

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such that

- each $f|_{E_i}$ is “simple”,
- the “size” of $G$ is $< \alpha$,
- the “simplicity” of $f$ on each piece and the number $M$ of pieces depends only on $\alpha$ (and maybe some ambient dimensions), but not on the particular map $f$. 

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Theorem (Federer?)

If $f : \mathbb{R}^d \to \mathbb{R}^n$ is Lipschitz, then there are sets $E_i \subseteq \mathbb{R}^d$ such that

(i) each $f|_{E_i}$ is bi-Lipschitz, and
(ii) $H_d(\cup E_i) = 0$.

Main idea: Using Rademacher’s theorem, choose $E_i$ to be sets on which $Df$ is “approximately constant.”

But this result is not quantitative: no control on number of pieces $E_i$, which is infinite, or the bi-Lipschitz constant of $f$ on each piece.
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Let $f : [0,1]^d \to Y$ be a $1$-Lipschitz map into an arbitrary metric space.
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Let $f : [0, 1]^d \to Y$ be a 1-Lipschitz map into an arbitrary metric space. Let $\alpha > 0$. 
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where

- each $f|_{E_i}$ is $L$-bi-Lipschitz,
- the $d$-dimensional Hausdorff content $\mathcal{H}^d_\infty(f(G)) < \alpha$,
- the bi-Lipschitz constant $L$ and the number $M$ of pieces depends only on $\alpha$ and $d$, but not on the particular map $f$ or space $Y$. 
Aside: metric domains

Schul’s theorem allows an arbitrary metric space as the **target** but requires a Euclidean **domain**.
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- similar results for mappings between Carnot groups (Meyerson ’13, Li ’15),
- counterexamples in many other settings due to David-Semmes, Laakso, Le Donne-Li-Rajala.
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The remainder of the talk will concern Euclidean domains and metric space targets, although the results are new even for Euclidean targets.
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 Decreasing dimension

Going down in dimension

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Question

What can we say for Lipschitz mappings that lower dimension? e.g., from $[0, 1]^3$ to $\mathbb{R}^2$?

What does “simple” mean in this context?
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Idea: “simple” should mean “behaves like an orthogonal linear projection from \( \mathbb{R}^{n+m} \) to \( \mathbb{R}^n \).”

i.e. if you write points of \( \mathbb{R}^{n+m} \) as \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \), we would like our map to look like

\[ (x, y) \mapsto x. \]
“Hard Sard sets”

Let \( f : [0, 1]^{n+m} \to Y \) be Lipschitz.
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Let $f : [0, 1]^{n+m} \to Y$ be Lipschitz.

**Definition (Azzam-Schul ’12)**

We say that $E \subseteq [0, 1]^{n+m}$ is a “**Hard Sard set**” for $f$ if

1. $F|_{g(E)}$ is constant on “vertical $m$-planes”: $(\{x\} \times \mathbb{R}^m) \cap g(E)$,
2. $F|_{g(E)}$ is bi-Lipschitz on “horizontal $n$-planes”: $(\mathbb{R}^n \times \{y\}) \cap g(E)$. 
Let $f : [0, 1]^{n+m} \to Y$ be Lipschitz.

**Definition (Azzam-Schul ’12)**

We say that $E \subseteq [0, 1]^{n+m}$ is a **“Hard Sard set”** for $f$ if there is a globally bi-Lipschitz map $g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ such that, if we let

$$F = f \circ g^{-1}$$

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Diagram of a “Hard Sard set”

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- (ii) the $n$-dimensional Hausdorff content $H^n(f(G)) < \alpha$,
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Answer: No!
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**Answer:** No!
Kaufman’s example

Theorem (Kaufman ’79)

There is a $C^1$ (hence Lipschitz) surjection $f : [0, 1]^3 \to [0, 1]^2$ such that $Df$ has rank $\leq 1$ everywhere.
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In particular, $f$ has no **Hard Sard set** on which it looks a projection from $\mathbb{R}^3 \rightarrow \mathbb{R}^2$, even though $\mathcal{H}^2_\infty(f([0,1]^3)) > 0$. 
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Note that Sard’s theorem says that Kaufman’s example cannot be made $C^2$. 
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Note that Sard’s theorem says that Kaufman’s example cannot be made $C^2$. 
The “mapping content”

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**Definition (Azzam-Schul ’12)**

Let $f : [0, 1]^{n+m} \to Y$ be Lipschitz and $A \subseteq [0, 1]^{n+m}$. The $(n, m)$-**mapping content** of $(f, A)$ is
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\mathcal{H}^{n,m}_\infty(f, A) = \inf \sum \mathcal{H}_\infty^n(f(Q_i))\text{side}(Q_i)^m
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where \( \{Q_i\} \) is a collection of dyadic cubes covering \( A \).
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- a bunch of cubes whose \( n \)-dimensional contents are compressed by \( f \):
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Azzam-Schul result

Let $f : [0, 1]^{n+m} \to Y$ be $1$-Lipschitz.

Azzam-Schul were able to use mapping content to find one big Hard Sard set of a given mapping $f$, i.e., where $f$ looks like a projection:
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**Theorem (Azzam-Schul ’12)**

Suppose $\mathcal{H}^n(Y) \leq 1$. ($Y$ is “$n$-dimensional”.)

Then $f$ has a Hard Sard set $E$ with

$$|E| \gtrsim 1$$

if and only if

$$\mathcal{H}^{n,m}_{\infty}(f, [0, 1]^{n+m}) \gtrsim 1,$$

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However, they did not completely decompose the domain of $f$ (up to arbitrarily small error) into pieces on which it looks like a projection.
Main question and answer

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**Question (Azzam-Schul ’12)**

Suppose $\mathcal{H}^n(Y) \leq 1$ and let $\alpha > 0$. 

**Theorem (GCD-Schul ’20)**

Yes.
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Question (Azzam-Schul '12)

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- the $(n, m)$-mapping content $\mathcal{H}_{\infty}^{n,m}(f, G) < \alpha$, 

Main question and answer

Let $f : [0, 1]^{n+m} \to Y$ be 1-Lipschitz.

**Question (Azzam-Schul '12)**

Suppose $\mathcal{H}^n(Y) \leq 1$ and let $\alpha > 0$. Can we write

$$[0, 1]^{n+m} = E_1 \cup E_2 \cup \cdots \cup E_M \cup G$$

where

- each $E_i$ is a Hard Sard set for $f$,
- the $(n, m)$-mapping content $\mathcal{H}_\infty^{n,m}(f, G) < \alpha$,
- $M$ and the Hard Sard constants depend only on $\alpha, n, m$?
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*where*

- each \( E_i \) is a Hard Sard set for \( f \),
- the \((n, m)\)-mapping content \( \mathcal{H}_{\infty}^{n,m}(f, G) < \alpha \),
- \( M \) and the Hard Sard constants depend only on \( \alpha, n, m \)?

**Theorem (GCD-Schul ’20)**

Yes.
Ideas behind the proof: Step 1

Let $f : [0, 1]^{n+m} \to Y$ be 1-Lipschitz, with $\mathcal{H}^n(Y) \leq 1$. 
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The proof of our main decomposition theorem breaks into two main steps.
Ideas behind the proof: Step 1

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The proof of our main decomposition theorem breaks into two main steps.

**Proposition (GCD-Schul '20)**

*If $F \subseteq [0, 1]^{n+m}$ is a set on which the map*

$$(x, y) \mapsto (f(x, y), y)$$

*is bi-Lipschitz, then $F$ can be quantitatively decomposed into Hard Sard sets, plus a set of small measure.*
Ideas behind the proof: Step 2

Let \( f : [0, 1]^{n+m} \to Y \) be 1-Lipschitz.
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Let $f : [0, 1]^{n+m} \to Y$ be 1-Lipschitz.

Proposition (GCD-Schul '20; closely related to David-Semmes '00)

Given $\alpha > 0$, we can write

$$[0, 1]^{n+m} = F_1 \cup \cdots \cup F_N \cup G$$

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where

- on each \( F_i \) there is a bi-Lipschitz change of coordinates \( \phi_i \) such that
  \[
  (x, y) \mapsto (f \circ \phi_i^{-1}(x, y), y)
  \]
  is bi-Lipschitz on \( \phi_i(F_i) \),

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- the $(n, m)$-mapping content $\mathcal{H}_{n,m}^{\infty}(f, G) < \alpha$,
- the decomposition is quantitative.
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Let \( f : [0,1]^{n+m} \rightarrow Y \) be 1-Lipschitz.

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where

1. *on each* \( F_i \) *there is a bi-Lipschitz change of coordinates* \( \phi_i \) *such that*

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2. *the* \( (n,m) \)-mapping content \( \mathcal{H}_{n,m}^{\infty}(f, G) < \alpha \),

3. *the decomposition is quantitative.*

Actually, this step works without any \( n \)-dimensionality assumption on \( Y \).
Ideas behind the proof: Quantitative Differentiation

“Quantitative Differentiation”: A Lipschitz map looks linear at “most” scales, not just infinitesimal ones.
Ideas behind the proof: Quantitative Differentiation

“Quantitative Differentiation”: A Lipschitz map looks linear at “most” scales, not just infinitesimal ones.

Theorem (Azzam-Schul ’14)

Let $f : [0, 1]^{n+m} \to Y$ be 1-Lipschitz and $\epsilon > 0$.

$\sum |Q|$ \lesssim \epsilon, n, m$.  

If $Y = \mathbb{R}^n$, “$\epsilon$-close to linear on $Q$” means that there is a linear map $A$ with $|f(x) - A(x)| < \epsilon$ side$(Q)$ for all $x \in Q$.  

(For mappings to metric spaces, one has to interpret “close to linear” appropriately.)
Ideas behind the proof: Quantitative Differentiation

“Quantitative Differentiation”: A Lipschitz map looks linear at “most” scales, not just infinitesimal ones.

**Theorem (Azzam-Schul ’14)**

Let $f : [0, 1]^{n+m} \rightarrow Y$ be 1-Lipschitz and $\epsilon > 0$. Then the set of dyadic cubes $Q$ on which $f$ is not $\epsilon$-close to linear satisfies

$$\sum |Q| \lesssim_{\epsilon, n, m} 1.$$
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Ideas behind the proof: Supplementing by projections to make bi-Lipschitz maps
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To prove our theorem, we want to find a set $F$ and a bi-Lipschitz change of coordinates $\phi$ such that

$$(x, y) \mapsto (f \circ \phi^{-1}(x, y), y)$$

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$F$ will be assembled from dyadic cubes. From our perspective, there are three types of cubes:
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$F$ will be assembled from dyadic cubes. From our perspective, there are three types of cubes:

1. $Q$ such that $f|_Q$ is far from linear.
2. $Q$ such that $f|_Q$ is close to a linear map of rank $< n$.
3. $Q$ such that $f|_Q$ is close to a linear map of rank $\geq n$.

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Quantitative Lipschitz decompositions

May 12, 2020 18 / 22
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Let’s therefore pretend that only type (3) cubes occur: cubes on which $f$ is close to a linear map of rank $\geq n$. 
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Let’s therefore pretend that only type (3) cubes occur: cubes on which \( f \) is close to a linear map of rank \( \geq n \).
If \( f \) was always close to the linear map \((x, y) \mapsto x\) on these cubes, then we could supplement \( f \) by \((x, y) \mapsto (f(x, y), y)\) and get a bi-Lipschitz map.
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Let’s therefore pretend that only type (3) cubes occur: cubes on which $f$ is close to a linear map of rank $\geq n$. If $f$ was always close to the *same* linear map, say $(x, y) \mapsto y$, then we could pre-compose $f$ by a rotation $\phi$, and then supplement by $(x, y) \mapsto (f \circ \phi^{-1}(x, y), y)$. 
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If $f$ was always close to the same linear map, say $(x, y) \mapsto y$, then we could pre-compose $f$ by a rotation $\phi$, and then supplement by $(x, y) \mapsto (f \circ \phi^{-1}(x, y), y)$.
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The issue is that, in principle, $f$ may switch between $(x, y) \mapsto x$ and $(x, y) \mapsto y$ arbitrarily many times as we zoom in.
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After this, one can form the change of coordinates $\phi$ by assembling different rotations at different scales, like a clockwork mechanism.
Thanks