GSE statistics without spin

joint work with

Chris Joyner and Martin Sieber

Sebastian Müller
Spectral statistics

Spectra of chaotic systems have statistics in agreement with random matrix theory. (Bohigas, Giannoni, Schmit 1984)

Ensemble depends on symmetries. Symmetry operators have to leave transition amplitudes $|\langle \phi | \psi \rangle|^2$ invariant. Unitary symmetries, e.g. geometrical anti-unitary symmetries: (generalised) time reversal invariance $T (|\phi \rangle + |\psi \rangle) = |\phi \rangle + |\psi \rangle^*$, $\langle T \phi | T \psi \rangle = \langle \phi | \psi \rangle^*$ physically we also need $T^2 |\psi \rangle = c |\psi \rangle$ together with anti-unitarity this implies $T^2 = \pm 1$

Example: $H = \hat{p}^2 / 2m + V(x)$ is invariant under complex conjugation $K$ with $K^2 = 1$
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Random matrix ensembles

- Gaussian Unitary Ensemble (no time-reversal invariance)
- Gaussian Orthogonal Ensemble (time-reversal invariance with $T^2 = 1$)
- Gaussian Symplectic Ensemble (time-reversal invariance with $T^2 = -1$)
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(in absence of unitary symmetries)
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Spin systems

E.g.: spin 1/2 system with spin-orbit coupling

\[ H = \hat{p}^2/2m + V(x) + \hbar^2/3 \sum_{i=1}^{3} \sigma_i L_i \]

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

commutes with \[ T = i \sigma_2 \]

\[ K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

where \[ T^2 = -1 \]

GSE statistics!
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GSE statistics!
Kramer’s degeneracy

for $T_2 = -1$ implies that states $|n\rangle$ and $|T_n\rangle$ are orthogonal and have the same energy. Write the Hamiltonian in a basis $|n\rangle$, $|T_n\rangle$:

$$H_{nm} = (\langle n|H|m\rangle \langle n|H|T_m\rangle \langle T_n|H|m\rangle \langle T_n|H|T_m\rangle)$$

It becomes quaternion-real, i.e.

$$H_{nm} = (\alpha \beta - \beta^* \alpha^*) = a_0 1 + a_1 i \sigma_1 = I + a_2 i \sigma_2 = J + a_3 i \sigma_3 = K$$
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for $T^2 = -1$ implies that states $|n\rangle$ and $|Tn\rangle$ are orthogonal and have same energy.

write Hamiltonian in a basis $|n\rangle$, $|Tn\rangle$: 

$$H_{nm} = \left( \langle n| H |m\rangle \langle n| H |Tm\rangle - \langle Tn| H |m\rangle \langle Tn| H |Tm\rangle \right)$$

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GSE statistics can arise without spin. Example: a quantum graph with discrete geometrical symmetries.
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Main message

GSE statistics can arise without spin.

- example: a quantum graph
GSE statistics can arise without spin.

- example: a quantum graph

- background: discrete geometrical symmetries
Quantum graphs

networks of vertices connected by bonds (with lengths)

Schrödinger equation on each bond

\[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}\psi(x) = E\psi(x)\]

conditions at the vertices:

- e.g. continuity
- Neumann conditions (sum over \(d\psi/dx\) of adjacent bonds is 0)

large well connected graphs display RMT spectral statistics

if Hamiltonian and vertex conditions symmetric w.r.t. complex conjugation: GOE
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Quantum graphs

[Complex Phase Factor] GUE
Quantum graphs

- Here time-reversal invariance is broken by a complex phase factor: GUE
Quantum chaos
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($P = \text{switching to other copy}, \ K = \text{complex conjugation}$)

$\mathcal{T}^2 = 1 \implies \text{GOE}$
Quantum graphs

The following graph has the anti-unitary symmetry $T$ defined by

$$T \psi(x) = \begin{cases} 
\psi^*(Px) & x \in \text{left half} \\
-\psi^*(Px) & x \in \text{right half}
\end{cases}$$

$$i\frac{\hbar}{2}T^2 = -\hbar^2 = \Rightarrow \text{GSE}$$

proposed realization: e.g. optical fibres
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General approach to symmetries
Symmetries

Spectral statistics in systems with (discrete) geometric symmetries

Example: reflection symmetry
- Two subspectra:
  - Eigenfunctions even under reflection ⇒ GOE
  - Eigenfunctions odd under reflection ⇒ GOE
- Subspectra uncorrelated
Symmetries

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Example: reflection symmetry
Symmetries

Spectral statistics in systems with (discrete) **geometric symmetries**?

**Example: reflection symmetry**

![Diagram of a reflection symmetric shape with two subspectra, one even and one odd under reflection, indicating GOE behavior.](image)
Symmetries

Spectral statistics in systems with (discrete) **geometric symmetries**?

**Example: reflection symmetry**

<table>
<thead>
<tr>
<th><img src="image_url" alt="Diagram" /></th>
</tr>
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Two subspectra:
Symmetries

Spectral statistics in systems with (discrete) geometric symmetries?

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- eigenfunctions even under reflection
Spectral statistics in systems with (discrete) **geometric symmetries**?

**Example: reflection symmetry**

![Diagram showing reflection symmetry with two subspectra: one for eigenfunctions even under reflection leading to GOE.](image-url)
Symmetries

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two subspectra:

- eigenfunctions even under reflection $\Rightarrow$ GOE
- eigenfunctions odd under reflection $\Rightarrow$ GOE
- subspectra uncorrelated
General discrete symmetries

- group of classical symmetry operations $g$
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- group of *classical* symmetry operations $g$
  
  in our example identity and reflection
General discrete symmetries

- group of **classical** symmetry operations \( g \)
  
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- **quantum** symmetries
  
  \[ U(g)\psi(r) = \psi(g^{-1}r) \]

  commute with Hamiltonian,
General discrete symmetries

- group of \textbf{classical} symmetry operations \( g \)

  in our example identity and reflection

- \textbf{quantum} symmetries

  \[ U(g)\psi(r) = \psi(g^{-1}r) \]

  commute with Hamiltonian,

  they form a representation of the classical symmetry group, i.e.,

  \[ U(gg') = U(g)U(g') \]
General discrete symmetries can diagonalize $H$ and block-diagonalize symmetry operators $U(g) = \begin{pmatrix} M_{T1}(g) & \ldots & M_{T1}(g) \\ \vdots & \ddots & \vdots \\ M_{T2}(g) & \ldots & M_{T2}(g) \end{pmatrix}$ blocks $M_\alpha(g)$ are (irreducible) matrix representations of the classical group, they satisfy $M_\alpha(gg') = M_\alpha(g)M_\alpha(g')$ eigenfunctions corresponding to each block have same energy if they are grouped into a vector $\psi$ we get: $U(g)\psi = M_\alpha(g)^T\psi$ consider subspectra corresponding to irreducible representations
General discrete symmetries

- can diagonalize $H$ and **block-diagonalize** symmetry operators
General discrete symmetries

- can diagonalize $H$ and **block-diagonalize** symmetry operators

$$U(g) = \begin{pmatrix} M_1^T(g) & & \\ & \ddots & \\ & & M_1^T(g) \end{pmatrix}$$

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$$M_\alpha(g g') = M_\alpha(g) M_\alpha(g')$$

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General discrete symmetries

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&M_1^T(g) \\
&\ddots & M_1^T(g) \\
& & M_1^T(g) & M_2^T(g) \\
& & & \ddots \\
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& \ddots \\
& & M_2^T(g)
\end{pmatrix}
\ldots
$$

- blocks $M_\alpha(g)$ are (irreducible) matrix representations of the classical group, they satisfy

$$M_\alpha(gg') = M_\alpha(g)M_\alpha(g')$$
can diagonalize $H$ and **block-diagonalize** symmetry operators

$$U(g) = \begin{pmatrix}
M_1^T(g) \\
\vdots \\
M_1^T(g) \\
M_2^T(g) \\
\vdots \\
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\end{pmatrix}$$

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eigenfunctions corresponding to each block have same energy
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- consider subspectra corresponding to irreducible representations
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$$
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& \ddots & \\
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$$

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- types of representations:
General discrete symmetries

- types of representations:
  - complex $M_\alpha$
General discrete symmetries

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- complex $M_\alpha$
- real $M_\alpha$
types of representations:

- complex $M_{\alpha}$
- real $M_{\alpha}$
- quaternion real (pseudo-real) $M_{\alpha}$
Statistics inside subspectra

Why?

consider \( T = \text{complex conjugation} \); 2d pseudo-real representation

\[ \psi \psi \psi \text{ transform according to } U(g) \psi \psi \psi = M_\alpha(g) T \psi \psi \psi \]

but \( T \psi \psi \psi \) transforms with \( (M_\alpha(g) T \psi \psi \psi)^* \) \( \Rightarrow \) \( T \) not compatible with structure of subspace

use \( \bar{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) \( T \) instead:

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\( \bar{T}^2 = -1 \) \( \Rightarrow \) GSE

Find a graph whose symmetry group has a pseudo-real representation.
Statistics inside subspectra

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Why?
Consider $T = \text{complex conjugation}$; 2d pseudo-real representation $\psi$ transforms according to $U(g)\psi = \mathcal{M}_\alpha(g)T\psi$ but $T\psi$ transforms with $(\mathcal{M}_\alpha(g)T)\psi$ $\Rightarrow T$ not compatible with structure of subspace.

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Find a graph whose symmetry group has a pseudo-real representation.
Construction of a GSE quantum graph

The simplest group with a pseudo-real representation is the quaternion group $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = ijk = -1 \}$.

Elements can be written as products of the generators $i$ and $j$.

Cayley graph: Group elements as vertices, bonds of length connect group elements related by right multiplication with bonds of length connect group elements related by right multiplication with...
Construction of a GSE quantum graph

simplest group with a pseudo-real representation:

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Cayley graph:

graph symmetric w.r.t. left multiplication with any group element
Construction of a GSE quantum graph
Construction of a GSE quantum graph

- increase size:
Construction of a GSE quantum graph

- increase size: replace vertices by sub-graphs
Construction of a GSE quantum graph

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graph with GSE subspectrum
Construction of a GSE quantum graph

... but boundary conditions mix pairs of degenerate eigenfunctions.
Construction of a GSE quantum graph

- take fundamental domain (eighth of graph)
Construction of a GSE quantum graph

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Construction of a GSE quantum graph

- take fundamental domain (eighth of graph)

and choose boundary conditions selecting GSE subspectrum
Construction of a GSE quantum graph

- take fundamental domain (eighth of graph)

![Graph diagram]

and choose boundary conditions selecting GSE subspectrum

\[
\begin{align*}
I &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{align*}
\]
Construction of a GSE quantum graph

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graph with pure GSE statistics
Construction of a GSE quantum graph

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![Graph Image]

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![Matrix Image]

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Construction of a GSE quantum graph
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- let each of the two eigenfunctions live on a separate copy of the graph
Construction of a GSE quantum graph

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Construction of a GSE quantum graph

Let each of the two eigenfunctions live on a separate copy of the graph.

Graph with a pure GSE spectrum and no resemblance of spin.
Numerical Results

1 + 1 + \frac{i}{1} - \frac{i}{1} - \frac{i}{1} - \frac{i}{1} - \frac{i}{1}
Numerical Results

\[ 1 + 1 + -1 -1 + i + i - i - i \]

\[
\begin{array}{ccccccc}
0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 & 1.2 \\
0 & 0.5 & 1 & 1.5 & 2 & 2.5 & 3 \\
\end{array}
\]

\[ P(s) \]

Agreement with GSE,
Agreement with GSE 😊
Conclusions

Discrete symmetries with pseudo-real representations can be used to generate GSE statistics. Quantum graph with $Q8$ symmetry has GSE subspectrum, this can be isolated. Generalisation to the 'tenfold way'? Experimental realisation?
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