Dependent Random Choice

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Marston Morse Lecture Series
October 26, 2016
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Example: Lower bound on Ramsey numbers

Definition

The Ramsey number $r(n)$ is the minimum $N$ such that every 2-edge-coloring of $K_N$ contains a monochromatic $K_n$.

Theorem (Erdős 1947)

$$r(n) > 2^{n/2}$$
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**Proof:** Color every edge of $K_N$ with $N = 2^{n/2}$ randomly. A $K_n$ is monochromatic with probability $2 \cdot 2^{-n(n-1)/2}$. Therefore, there is a coloring without any monochromatic cliques.

**Open Problem:** Find an explicit coloring giving $r(n) > 2^{cn}$. Progress: Frankl-Wilson, Barak-Rao-Shaltiel-Wigderson, Chattopadhyay-Zuckerman, Cohen.
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$\exists$ a monochromatic $K_n$ with probability at most $\binom{N}{n} 2 \cdot 2^{-\binom{n}{2}} < 1$. 

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\( \exists \) a monochromatic \( K_n \) with probability at most \( \left( \frac{N}{n} \right) 2 \cdot 2^{-\binom{n}{2}} < 1 \).

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**Proof idea:** Let $U$ be the set of vertices adjacent to every vertex in a random set $R$ of appropriate size.
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As \( G \) is dense, we expect \( U \) to be large.
Dependent Random Choice

**Rough Claim**

Every *dense* graph $G$ contains a large vertex subset $U$ in which every set of $d$ vertices has many common neighbors.

**Proof idea:** Let $U$ be the set of vertices adjacent to every vertex in a random set $R$ of appropriate size.

As $G$ is dense, we expect $U$ to be large.

If $d$ vertices have only few common neighbors, it is very unlikely that $R$ will be chosen among these neighbors. Hence we do not expect $U$ to contain any such $d$ vertices.
Dependent Random Choice has many applications in Ramsey Theory, Extremal Graph Theory, Additive Combinatorics, and Combinatorial Geometry. Example we will cover include:

- Erdős problem on heavy monochromatic cliques
- Conjectures of Hajós and Erdős-Fajtlowicz
- Conjectures of Erdős-Simonovits and Sidorenko
- Burr-Erdős conjecture on Ramsey numbers
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**Definition**

Let $f(N)$ be the maximum number such that every 2-coloring of the edges of the complete graph on $\{2, \ldots, N\}$ has a monochromatic clique of weight at least $f(N)$. 

Conjecture: (Erdős 1981) $f(N) \to \infty$.

Problem: (Erdős 1981) Estimate $f(N)$.

Theorem: (Rödl 2003) $c_1 \log \log \log \log N \log \log \log \log N \leq f(N) \leq c_2 \log \log \log N$. 
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\[ c_1 \frac{\log \log \log \log N}{\log \log \log \log \log N} \leq f(N) \leq c_2 \log \log \log N. \]
Upper bound proof: \( f(N) = O(\log \log \log N) \)
Partition vertex set \([2, N]\) into \(s = \log \log N\) intervals
\(I_j = [2^{2j-1}, 2^{2j})\).
Upper bound proof: \( f(N) = O(\log \log \log N) \)

Partition vertex set \([2, N]\) into \( s = \log \log N\) intervals
\( I_j = [2^{2^{j-1}}, 2^{2^j}) \).

Color edges inside interval \( I_j \) with no monochromatic set of order
\( 2 \log 2^{2^j} = 2^{j+1} \). Then \( I_j \) contributes at most \( 2^{j+1}/\log 2^{2^{j-1}} = 4 \) to
the weight of any monochromatic clique.
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Consider a 2-edge-coloring $c$ of the complete graph on $[s]$ with no monochromatic $K_t$ with $t > 2\log s$. For $j \neq j'$, color the edges from $I_j$ to $I_{j'}$ the color $c(j, j')$. 
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Partition vertex set \([2, N]\) into \( s = \log \log N \) intervals 
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\( 2 \log 2^{2^j} = 2^{j+1} \). Then \( I_j \) contributes at most \( 2^{j+1} / \log 2^{2^j-1} = 4 \) to the weight of any monochromatic clique.

Consider a 2-edge-coloring \( c \) of the complete graph on \([s]\) with no monochromatic \( K_t \) with \( t > 2 \log s \). For \( j \neq j' \), color the edges from \( I_j \) to \( I_{j'} \) the color \( c(j, j') \).

Hence,
\[
f(N) \leq 4 \cdot 2 \log s = 8 \log \log \log N.
\]
Theorem: (Conlon-F.-Sudakov)

\[ f(N) = \Theta(\log \log \log N). \]

That is, every 2-edge-coloring of the complete graph on \( \{2, \ldots, N\} \) contains a monochromatic clique \( S \) with

\[ \sum_{i \in S} \frac{1}{\log i} = \Omega(\log \log \log N). \]
A subdivision of a graph is obtained by replacing edges by paths.
Hajós conjecture

Conjecture: (Hajós 1961)
If a graph contains no subdivision of $K_t + 1$, then it is $t$-colorable.

Strengthening of Hadwiger's conjecture and the Four Color Theorem.

Disproved by Catlin in 1979 for $t \geq 6$.

Erdős and Fajtlowicz in 1981 showed that: almost all graphs are counterexamples!
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Erdős-Fajtlowicz theorem

\[ \sigma(G) = \text{maximum } t \text{ for which } G \text{ contains a subdivision of } K_t. \]
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**Theorem: (Erdős and Fajtlowicz 1981)**

The random graph \( G = G(n, 1/2) \) almost surely satisfies

\[ \chi(G) = \Theta\left(\frac{n}{\log n}\right) \quad \text{and} \quad \sigma(G) = \Theta(\sqrt{n}). \]
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\( H(n) \) is the maximum of \( \chi(G)/\sigma(G) \) over all \( n \)-vertex graphs \( G \).
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**Theorem: (Erdős and Fajtlowicz 1981)**
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Hajós conjectured \( H(n) = 1. \)
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They further conjectured that
the random graph is essentially the strongest counterexample!
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**Conjecture:** (Erdős and Fajtlowicz 1981)

There is \( C \) such that for all \( n \),

\[ H(n) < Cn^{1/2}/\log n. \]
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**Theorem: (F.-Lee-Sudakov)**

The Erdős-Fajtlowicz conjecture is true.
Ramsey Theory

Large systems contain patterns.

Question:

How many monochromatic copies of a graph $H$ must there be in every 2-edge-coloring of $K_n$?

Conjecture: (Erdős 1962, Burr-Rosta 1980)

For each $H$, the random 2-edge-coloring of $K_n$ in expectation asymptotically minimizes the number of monochromatic copies of $H$ over all 2-edge-colorings of $K_n$.

Random systems minimize patterns.

(Goodman 1959) True for $H = K_3$.

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(Random systems minimize patterns. (Goodman 1959) True for $H = K_3$. (Thomason 1989) False for $H = K_4$.)
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Known for trees, complete bipartite graphs, even cycles, and cubes.
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Has connections to matrix theory, Markov chains, graph limits, and quasi-randomness.
**Homomorphism version**

**Definition:**

\[ h(H) = \text{number of homomorphisms from } H \text{ to } G. \]

\[ t(H)(G) = \frac{h(H)(G)}{|G||H|} = \text{fraction of mappings from } H \text{ to } G \text{ which are homomorphisms.} \]

**Conjecture:** (Sidorenko, Erdős-Simonovits 1980s)

For every bipartite graph \( H \) and every graph \( G \),

\[ t(H)(G) \geq e(H) t(K_2)(G) \]

where \( e(H) \) is the number of edges in graph \( H \).
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**Conjecture: (Sidorenko, Erdős-Simonovits 1980s)**
For every bipartite graph \( H \) and every graph \( G \),
\[
t_H(G) \geq t_{K_2}(G)^{e(H)}.
\]
Let $\mu$ be the Lebesgue measure on $[0, 1]$, 

- Remark: The expression on the left hand side is quite common. For example, Feynman integrals in quantum field theory, Mayer integrals in statistical mechanics, and multicenter integrals in quantum chemistry are of this form.
Let $\mu$ be the Lebesgue measure on $[0, 1]$, $w(x, y)$ be a bounded, non-negative, symmetric and measurable function on $[0, 1]^2$, then

$$\int \prod_{(i,j) \in E} w(x_i, y_j) \, d\mu \geq (\int w \, d\mu)^2 |E|.$$
Analytic formulation of Sidorenko’s Conjecture

Let $\mu$ be the Lebesgue measure on $[0, 1]$, $w(x, y)$ be a bounded, non-negative, symmetric and measurable function on $[0, 1]^2$, and $E$ be a subset of $[t] \times [s]$.

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\int \prod_{(i,j) \in E} w(x_i, y_j) \, d\mu^{s+t} \geq \left( \int w \, d\mu^2 \right)^{|E|}
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Quasirandom graphs

Chung, Graham, and Wilson: a large number of interesting graph properties satisfied by random graphs are all equivalent.

**Definition**

A sequence \((G_n : n = 1, 2, \ldots)\) of graphs is called quasirandom with density \(p\) if, for every graph \(H\),

\[
\tau_H(G_n) = p e(H) + o(1).
\]

(1)

One of the many equivalent properties is that every subset \(S\) contains \(p(|S|^2) + o(n^2)\) edges.

Surprising fact

Quasirandomness follows from (1) for \(H = K_2\) and \(H = C_4\).
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One of the many equivalent properties is that every subset \(S\) contains \(p(\frac{|S|}{2}) + o(n^2)\) edges.
Chung, Graham, and Wilson: a large number of interesting graph properties satisfied by random graphs are all equivalent.

**Definition**

A sequence \((G_n : n = 1, 2, \ldots)\) of graphs is called *quasirandom with density* \(p\) if, for every graph \(H\),

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t_H(G_n) = p^{e(H)} + o(1).
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**Surprising fact**

Quasirandomness follows from (1) for \(H = K_2\) and \(H = C_4\).
A graph $F$ is $p$-forcing if $t_H(G_n) = p^{e(H)} + o(1)$ holds for $H = K_2$ and $H = F$ implies $(G_n)$ is quasirandom with density $p$. 

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- Which graphs are forcing?
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Hence, the forcing conjecture holds for a large class of graphs.
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\( r(H) \) is the minimum \( N \) such that every 2-edge-coloring of \( K_N \) contains a monochromatic copy of graph \( H \).
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2^{n/2} \leq r(K_n) \leq 2^{2n}.
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How large is \( r(H) \) for a sparse graph \( H \) on \( n \) vertices?
Conjecture (Burr-Erdős 1975)

For every $d$ there is a constant $c_d$ such that if a graph $H$ has $n$ vertices and maximum degree $d$, then

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If $H$ has $n$ vertices and maximum degree $\Delta$, then $r(H) \leq c(\Delta, k)n$.

Proved using the hypergraph regularity method for $k=3$ by Cooley-Fountoulakis-Kühn-Osthus and Nagle-Rödl-Olsen-Schacht, and for all $k$ by CFKS. Gives Ackermann-type bound on $c(\Delta, k)$.

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