Positive entropy automorphisms of varieties

- $\phi : X \to X$ a self-map of a compact metric space.

- Entropy = “how fast does $\phi$ separate points of $X$?”
  - $N(n, \epsilon) =$ maximum size of set of points such that any two have $d(f^k(p_1), f^k(p_2)) \geq \epsilon$ for some $0 \leq k \leq n$.

- $h_{\text{top}} = \lim_{\epsilon \to 0} \left( \limsup_{n \to \infty} \frac{\log N(n, \epsilon)}{n} \right)$

- Theorem (Gromov–Yomdin). For automorphisms of varieties over $\mathbb{C}$, entropy can be computed in cohomology.

- Non-example: $\phi \in \text{PGL}(3)$ acting on $\mathbb{P}^2$. 
An example (J. Blanc)

- Fix an elliptic curve $E \subset \mathbb{P}^2$, $p \in E$.

- Define $\sigma_p : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ to act on $L_{pxy}$ by the involution of $\mathbb{P}^1$ fixing $x$ and $y$.

- Not defined where the line is tangent to $E$!
To resolve $\sigma_p$, blow up $p$ and the four points $p_i$.

$\tilde{\sigma}_p : X_p \to X_p$ is an involution, fixing the curve $E$.

Same thing with different initial point $q$ gives $\tilde{\sigma}_q : X_q \to X_q$.

Both $\sigma_p$ and $\sigma_q$ lift to the common blow-up $X_{p,q}$.

$\tilde{\sigma}_p \circ \tilde{\sigma}_q$ is a positive entropy automorphism of $\text{Bl}_{10 \text{ pts}} \mathbb{P}^2$.

Many other constructions: McMullen, Bedford–Kim, . . .
It’s sometimes hard to find examples of “pathological” behaviors on high-dimensional varieties.

Varieties with interesting dynamics provide a good source: rich geometry, tractable computations.
Question: (Kawamata)

Fix a smooth projective variety $X$. Is the number of smooth projective varieties $Y$ with $D^b \text{Coh}(X) \cong D^b \text{Coh}(Y)$ finite?

- Yes, for many nice classes: curves, surfaces, abelian, toric, Fano, $K_X$ ample, . . .
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- No, for $X$ the blow-up of $\mathbb{P}^3$ at $\geq 8$ general points.

- The $Y$’s are all $\mathbb{P}^3$ blown up at different configurations of 8 points, arising from dynamics of the Cremona group $\text{Bir}(\mathbb{P}^3)$. 

Theorem: (De-Qi Zhang)

Suppose that $\phi : X \to X$ is a birational automorphism of infinite order. Then either:

- $\phi$ is imprimitive: $0 < \dim B < \dim X$ and

  $\begin{array}{ccc}
  X & \to & X \\
  \downarrow & & \downarrow \\
  B & \to & B
  \end{array}$

- $X$ is birational to a weak Calabi-Yau or abelian variety;

- $X$ is rationally connected.
Question: (Bedford ’11)
Does there exist a blow-up of $\mathbb{P}^3$ that admits an automorphism of positive entropy?

Question: (Kawamata–Matsuda–Matsuki ’87)
Does there exist a variety of non-negative Kodaira dimension with infinitely many flipping curves?