Transversality of loop coproduct and cobracket

Dingyu Yang
Transversality for moduli spaces of holomorphic curves

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- \((W^{2n}, \omega)\) possibly with ends like \((\mathbb{R}^\pm \times V, d(e^s \lambda))\)’s.
- \(L^n\) embedded Lagrangian possibly with ends like \((\mathbb{R}^\pm \times \text{Leg. } S)\)’s.
- auxilliary \(J\).
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$\Rightarrow$ moduli spaces of $J$-holo. curves in $(W, L)$ w/ bdries, punctures, levels. w/ evaluation map at punctures and (punctured) bdry loops.
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w/ evaluation map at punctures and (punctured) bdry loops.

Conjectural picture (Fukaya, C-L-M): codimensional-1 boundary (filtered)
= (fiber) product + loop bracket + loop cobracket of evaluation maps
from simpler moduli space(s).
transversality of these "intersection"-type operations on maps
nontrivial transversality of domain moduli space as solution of $\bar{\partial} J$, Fredholm problem with domain variation, analytic limiting behavior.

punctured bordered (broken) curves

$(\mathbb{R} \times V, \mathbb{R} \times S)$ symplectization Lag 

$\partial I$

(fiber) product Loop bracket (2 domains) Loop bracket (1 domain) Loop cobracket
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• nontrivial transversality of domain moduli space as solution of $\bar{\partial}J$, Fredholm problem with domain variation, analytic limiting behavior.
Coproduct at chain level is relevant

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loop product

loop coproduct
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Focus on coproduct, severe transversality issue. Joint w/ Manuel Rivera.
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$e_\varphi : U \times [0,1] \to M \times M, (x, t) \mapsto (\varphi(x)(0), \varphi(x)(t))$, $\dot{e}_\varphi := e_\varphi|_{U \times (0,1)}$.

Self-inters. spacetime is $e_\varphi^{-1}(\Delta_M)$, almost never good.
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But $\varphi \Rightarrow$ a chain $(\Theta_{\varphi}, \nu)$ by thickening domain s.t. $P_{\varphi} := (\hat{e}_{\Theta_{\varphi}})^{-1}(\Delta_M)$ is a manifold, killing directions by adjoining a Thom form $\nu$. 
Achieving transversality of loop coproduct via a thickening

- smooth $\alpha : [0, 1]/ \sim \to [0, 1]/ \sim$ is 0 on $[0, \epsilon] \cup [1 - \epsilon, 1]$, o/w diffeo.
- $\lambda_1 : [0, 1]/ \sim \to [0, 1]$ smooth $\lambda_1|_{[-\epsilon/2, \epsilon/2]} = ct$ and then cut-off.
- $\lambda_2 : [0, 1]/ \sim \to [0, 1]$ is 0 on $[-\epsilon/4, \epsilon/4]$ and positive o/w.
- A Riemannian metric on $M$ with inj.rad. $\delta$ and exp.
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Thickening: $\Theta_\varphi(x, v, w)(t) := \exp_{\varphi(x)(\alpha(t))}(\lambda_1(t)P_t^1(v) + \lambda_2(t)P_t^2(w)),$
where $(x, v, w) \in \varphi(\cdot)(0)^*D_\delta(TM) \oplus U \varphi(\cdot)(1/2)^*D_\delta(TM)$ and $P_t^1$ and $P_t^2$
parallel transports along loop $\varphi(x) \circ \alpha$ from 0 to t and 1/2 to t.
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Since $\Theta_{\varphi}(x, v, w)(0) = \varphi(x)(0)$, $\Theta_{\varphi} \pitchfork \Delta_M$.

For $t$ close to 0, self-intersection equation can be uniquely solved in $v$ as a graph which extends smoothly over to $t = 0$. $\Rightarrow$ The closure is a manifold!
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Expect: smooth $\gamma \mapsto (c_\gamma, \epsilon_\gamma)$, $\Theta_\varphi$ can be defined w/o stopping loops via $\alpha$. 

Co-$A_\infty$ for chain map modification, loop cobracket

Thom form $\nu$ for domain thickening depending only on $M$.

$V : (\varphi, \eta) \mapsto$

(a loop pair by splitting over $P_\varphi$, smoothened via $\alpha, \iota^* pr_1^*(\nu \wedge \pi^* \eta)$).
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- Evaluation of varying base point $s$ and time $t$ in the lower 2-simplex, thicken $\Rightarrow$ htpy for commutativity symmetrize $\Rightarrow$ loop cobraclacket.
de Rham chains and polyfold–Kuranishi correspondence

Fukaya, Irie (w/o corners) for loop product/bracket.
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More generally, /technicality, de Rham chains $C_k(LM)$ generated over $\mathbb{R}$ by $(\varphi : U \to LM, \eta)$, $U$ or $k_1$-mfd w/ corners, $\eta \in \Omega^{k_2}_{\text{cpt}}(U)$, $k = k_1 - k_2$: 

- Linearity of $\eta$,
- Sum for disjoint union of domain,
- (diff dims) $(\varphi \circ \pi, \eta) \sim (\varphi, \pi^! \eta)$ for surjective submersion $\pi$.

$\partial \pm (\partial + \text{d})$. Quasi-iso to singular chains, good for higher homotopies.

Domain "spaces" can be described smoothly by Hofer-Wysocki-Zehnder's Polyfold Fredholm structure or Fukaya-Oh-Ohta-Ono's Kuranishi structure.

My thesis, polyfold–Kuranishi correspondence, lifts Kuranishi theory to equivalence class level, show both frameworks are equivalent.

Ev map from domain structure $\Rightarrow$ (generalized) dR chains (loop spaces).
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Near-future goal: (Compatibility with BV $\Rightarrow$) non-marked point version of Irie’s loop bracket and above loop cobaracket on equivariant chains are involutive bi Lie algebra up to higher homotopy. $IBL_\infty$, C-F-L.

Eventual goal: Using $IBL_\infty$ structure to achieve transversality. Moduli spaces satisfying codimensional-1 degeneration $\Rightarrow$ Maurer-Cartan elements $\Rightarrow$ twisted (filtered) $IBL_\infty$. $\Rightarrow$ invariants for $((V, \ker \alpha), S)$, indep. of $J$, $\alpha$ up to $IBL_\infty$ htpy equiv.

Contains relative symplectic field theory (with genus and multiple boundary components), e.g. splitting a Fukaya algebra along a contact hypersurface.
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Thank you!