Filling metric spaces

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Theorem

(M. Gromov) Let $M^n$ be a closed manifold. Consider its (isometric) Kuratowski embedding in $L^\infty(M)$. Then $M^n$ bounds in its $c(n)\text{vol}^{\frac{1}{n}}(M)$-neighbourhood. In other terms,

$\text{FillRad}(M^n) \leq c(n)\text{vol}^{\frac{1}{n}}(M^n)$. 

Essential manifolds: A class of closed non-simply connected manifolds that includes all non-simply connected closed surfaces, $\mathbb{R}P^n$, aspherical manifolds (including tori), etc. $M^n$ is essential if the classifying map $f : M^n \to B\pi_1(M^n)$ satisfies $f^*([M^n]) \neq 0 \in H^n(B\pi_1(M^n))$. Here: $B\pi_1(M^n)$ is the aspherical space with the fundamental group $\pi_1(M^n)$; $f^*$ homomorphism of homology groups induced by $f$; if $M^n$ is non-orientable, one considers homology groups with $\mathbb{Z}_2$ coefficients; $[M^n]$ is the fundamental homology class.
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Corollary

\textit{(M. Gromov)} Let $M^n$ be an essential manifold. Then the length of the shortest non-contractible closed curve does not exceed $6 \text{FillRad}(M^n) \leq \text{const}(n) \text{vol}^{\frac{1}{n}}(M^n)$. 
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Proof (in case $M^n$ aspherical): Fill $M^n$ by chain $W^{n+1}$ in $\text{FillRad}(M^n)$-neighbourhood of $M^n$ in $L^\infty$. Proceed by contradiction. Assume not.
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Larry Guth: Two important advances solving two Gromov’s conjectures:

1. First, \( \text{FillRad}(M^n) \leq 1 \) follows already from the assumption that the volume of each metric ball of radius 1 is less than some constant \( \text{const}(n) \) rather than assuming \( \text{Vol}(M^n) \leq \text{const}(n) \).

2. The assumption that the volume of all balls of radius 1 are small yields more: One can conclude that the \((n-1)\)-dimensional Urysohn width of \( M^n \) is less than 1. Urysohn \((n-1)\)-dimensional width, \( \text{UW}_{n-1}(X) \), of \( X \): the infimum of \( d \) such that there exists a continuous map \( X \to \mathbb{K}_{n-1} \) such that \( \mathbb{K}_{n-1} \) is a \((n-1)\)-dim CW-complex, and for each \( y \in \mathbb{K}_{n-1} \) \( \text{diam}(F_{n-1}(y)) \leq d \). Informally: If volumes of all balls of radius 1 in \( M^n \) are sufficiently small, then \( M^n \) is close to a \((n-1)\)-dimensional CW-complex.
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Urysohn $(n-1)$-dimensional width, $\text{UW}_{n-1}$, of $X$: the infimum of $d$ such that there exists a continuous map $X \rightarrow K^{n-1}$ such that $K^{n-1}$ is a $(n-1)$-dim CW-complex, and for each $y \in K^{n-1}$ $\text{diam}(F^{-1}(y)) \leq d$. 
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(L. Guth). There exists $\epsilon_n > 0$ with the following property: Assume that each metric ball of radius 1 in a closed Riemannian manifold $M^n$ has volume less than $\epsilon_n$. Then the Urysohn $(n - 1)$-width of $M^n$ is less than or equal to 1.
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2. (Guth) Is it true that there exists $\epsilon_m > 0$ with the following property: Let $X$ be a compact metric space such that the $m$-dimensional Hausdorff content of each metric ball of radius 1 is less than $\epsilon_m$. Then $UW_{m-1}(X) \leq 1$. 

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Main result: Guth’s conjecture is true:

**Theorem**

(Y. Liokumovich, B. Lishak, A.N., R. Rotman) There exists a positive $\epsilon_m$ with the following property. Let $X$ be a compact (or even boundedly compact) metric space. Assume that for some positive $R \text{HC}_m(B) \leq \epsilon_m R^m$ for each metric ball $B$ of radius $R$ in $X$. Then $U\text{W}_{m-1}(X) \leq R$. 

**Corollary**

For each compact metric space $X$ $U\text{W}_m(X) \leq \text{const}(m)$ $\text{HC}_1(X)$. 

**Motivation(s):**

1. Even for Riemannian manifolds this gives an intrinsic metric criterion when an $n$-dimensional closed or complete non-compact manifold is close not merely to a $(n-1)$-dimensional CW-complex but a $(m-1)$-dimensional one for any $m \leq n$. 

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1. Even for Riemannian manifolds this gives an intrinsic metric criterion when a $n$-dimensional closed or complete non-compact manifold is close not merely to a $(n - 1)$-dimensional CW-complex but a $(m - 1)$-dimensional one for any $m \leq n.$
2. If $m$-dimensional Hausdorff measure ($\mathcal{X}$) $= 0$, then $HC_m(\mathcal{X}) = 0$. Now the corollary implies that $UW_{m-1}(\mathcal{X}) = 0$. Thus, the covering dimension of $\mathcal{X} \leq m - 1$. We obtained the Szpilrajn theorem.
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3. Even for $m$-dimenional Riemannian manifolds this is a strengthening of previous results by Gromov and Guth, as $HC_m(M^m) \leq vol(M^m)$, and, in fact, can be much smaller. (Think about hyperbolic disc of radius 1 but with a very large negative curvature.)
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Proof of Gromov’s theorem:
Pseudo-proof. Pretend that $L^\infty(M^n)$ is $\mathbb{R}^N$. Fill $M^n$ by a minimal surface $W^{n+1}$.

Isoperimetric inequality: $\text{vol}(W^{n+1}) \leq C \text{vol}(M^n)^{\frac{n+1}{n}}$.
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Monotonicity formula implies that $W^{n+1}$ is in the
$\sim \text{vol}(W)^{\frac{1}{n+1}} = const(n)\text{vol}(M^n)^{\frac{1}{n}}$-neighbourhood of $M^n$. 
S. Wenger’s version of Gromov’s proof of the isoperimetric inequality:
Based on 1) Coarea inequality; 2) Cone inequality.
Induction with respect to $n$.
The base (filling of closed curves) is easy by coning.
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Induction step:
Improve $M$: Cover a significant part of volume of $M$ by disjoint metric balls $B_i(r_i)$ in the ambient Banach space.
$\text{vol}(B_i \cup M) \geq \left(\frac{r_i}{1000}\right)^n$, and is an almost maximal among concentric balls with this property.
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\[ \text{vol}_{n-1}(\partial B_i \cap M) \leq \frac{r_i^{n-1}}{1000^n} \]
(Use coarea inequality).
Now cut $B_i$ out, replace by a “good” filling (that exists by induction assumption), and project the filling inside $B_i$. Fill the gap between $M = M_1$ and the “improved version” $M_2$ of $M_1$ by coning (in $B_1$).
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Proof of Guth’s theorem (case when $M^n$ is Riemannian manifold; $m = n$; $n$-dimensional Hausdorff measure = the volume instead of $HC_n$):

Guth:
1. Find a “good covering” of the manifold $M^n$ by “small” balls.
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3. Replace the image of the manifold in each face of the maximal dimension by the (almost) minimal surface with the same boundary. Monotonicity implies that this (almost) minimal surface is near the boundary of the face.
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$$\text{vol}(\phi(X) \cap \text{StarF}) \leq \text{const}(n)\epsilon_m r_i^m \exp(-\text{Const}(n) \text{ dim}(F)),$$

where $\phi$ is the map to the nerve, $F$ is a face, $r_1$ is the smallest radius of a ball in the intersection of metric balls corresponding to $F$. 
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Need: if $d$ “good” balls in the covering intersect, then the smallest radius behaves as $\exp(-\text{const}(n)d)$. 

By-product of Guth’s construction: If $B$ is a good ball of radius $r$, $\epsilon_n$ small, then

$$\text{vol}(B) \leq (r^{1000})^n (\text{in fact even} \leq (r^{1000})^{n+1}).$$
When we project, each time volume estimate gets multiplied by a constant \( L_d > 1 \). Yet we do not have any control over how many times we need to project. The product of all \( L_d \) must converge. Need:

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\text{vol}(\phi(X) \cap \text{Star}F) \leq \text{const}(n) \epsilon_m r_i^m \exp(-\text{Const}(n) \dim(F)),
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where \( \phi \) is the map to the nerve, \( F \) is a face, \( r_1 \) is the smallest radius of a ball in the intersection of metric balls corresponding to \( F \).

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$HC_m$. Main problem: Struggle against non-additivity of $HC_m$. 

Assume we have $B_1 \bigcup B_2$, where $B_i$ is a ball of radius $r_i$, and $r_i$ are comparable. Non-additivity: due an overlap in optimal coverings of $B_1$ and $B_2$. Idea: $HC_m(B_i) < r_i^{1000} m$ implies one need to use balls of radius $< r_{100}$ to cover $B_1 \bigcup B_2$. The sum of $m$th powers of radii is greater than $HC_m$ of $(1 - \frac{1}{100}) B_1 \bigcup (1 - \frac{1}{100}) B_2$. It remains to ensure that $HC_m$ of the somewhat smaller balls is comparable with a content of larger balls.
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Assume we have $B_1 \cup B_2$, where $B_i$ is a ball of radius $r_i$, and $r_i$ are comparable.
Non-additivity: due an overlap in optimal coverings of $B_1$ and $B_2$.
Idea: $HC_m(B_i) < (\frac{r_i}{1000})^m$ implies one need to use balls of radius $< \frac{r}{100}$ to cover $B_1 \cup B_2$. The sum of $m$th powers of radii is greater than $HC_m$ of $(1 - \frac{1}{100})B_1 \cup (1 - \frac{1}{100})B_2$. It remains to ensure that $HC_m$ of the somewhat smaller balls is comparable with a content of larger balls.
Lemma

(Co-area inequality) Let $U \subset B(R_2) \setminus B(R_1)$ be a closed set. Then,

$$\int_{R_1}^{R_2} HC_{m-1}(S_R \cap U) \, dR \leq 2HC_m(U).$$

Therefore, there exists $R \in [R_1, R_2]$, such that

$$HC_{m-1}(S_R \cap U) \leq \frac{2}{R_2 - R_1} HC_m(U).$$
Proof.

Let \( \{ B_{r_i}(p_i) \} \) be a covering of \( U \) with \( \sum_i r_i^m \leq HC_m(U) + \epsilon \), where \( i \in \{1, \ldots, N\} \) for some \( N \). The desired inequality would follow from the inequality \( \int_{R_1}^{R_2} HC_{m-1}(S_R \cap U) \ dR \leq 2 \sum_i r_i^m \). We are going to prove a stronger inequality, where \( HC_{m-1}(S_R \cap U) \) is replaced by the following quantity that is obviously not less than \( HC_{m-1}(S_R \cap U) \), namely, \( \sum_{i \in I(R)} r_i^{m-1} \), where \( I(R) \) denotes the set of all indices \( i \) such that the intersection of \( B_{r_i}(p_i) \) and \( S(R) \) is not empty. The left hand side of the desired inequality becomes

\[
\int_{R_1}^{R_2} \sum_{i \in I(R)} r_i^{m-1} dR = \int_{R_1}^{R_2} \sum_{i=1}^{N} r_i^{m-1} \chi_i(R) dR = \sum_{i=1}^{N} r_i^{m-1} \int_{R_1}^{R_2} \chi_i(R) dR,
\]

where the characteristic function \( \chi_i(R) \) is equal to 1 for all \( R \in [R_1, R_2] \) such that \( S_R \) and \( B_{r_i}(p_i) \) have a non-empty intersection, and to 0 otherwise. Finally, observe that \( \int_{R_1}^{R_2} \chi_i(R) dR \leq 2r_i \), which implies the desired inequality. \( \square \)