Topological and arithmetic intersection numbers attached to real quadratic cycles

Henri Darmon, McGill University

Jan Vonk, McGill University

Workshop, IAS, November 8
This is joint work with Jan Vonk
Preamble

Arithmetic quotients of symmetric spaces, and topological cycles on them, often behave “as if” they were algebraic.

For instance, a modular curve $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ is equipped with a canonical collection of:

- CM zero cycles, which are algebraic, and defined over ring class fields of imaginary quadratic fields.

- geodesic cycles attached to ideal classes of real quadratic quadratic fields, which are not algebraic.

**Claim**: These geodesic cycles (and their quaternionic analogues) encode the valuations of a richer collection of invariants, suitable for generating class fields of real quadratic fields.
Singular moduli

A *singular modulus* is a value of $j(z)$ at a quadratic imaginary argument (CM point) in the Poincaré upper half plane $\mathcal{H}$.

The *theory of complex multiplication* asserts that these values are *algebraic integers*.

Examples:

$$j(i) = 1728; \quad j\left(\frac{1+\sqrt{-3}}{2}\right) = 0; \quad j\left(\frac{1+\sqrt{-7}}{2}\right) = -3375.$$  

$$j\left(\frac{1+\sqrt{-23}}{2}\right) = w, \quad \text{where}$$

$$w^3 + 3491750w^2 - 5151296875w + 12771880859375 = 0.$$  

It generates the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$. 
Differences of singular moduli and their factorisations

**Gross, Zagier (1984).** For all $\tau_1, \tau_2$ quadratic imaginary, the quantity

$$J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2) \in H_{12} := H_{\tau_1}H_{\tau_2}$$

is a smooth algebraic integer with an explicit factorisation.

All the primes $q$ dividing $\text{Norm}J_\infty(\tau_1, \tau_2)$ are $\leq D_1 D_2/4$.

The valuation $\text{ord}_q \text{Norm}J_\infty(\tau_1, \tau_2)$ is related to the *topological intersection* of certain CM 0-cycles on a zero-dimensional Shimura variety, attached to the definite quaternion algebra ramified at $q$ and $\infty$. 
Let

\[ J_{\infty}(D_1, D_2) := \prod J_{\infty}(\tau_1, \tau_2), \quad \text{disc}(\tau_1) = D_1, \text{disc}(\tau_2) = D_2, \]

**Kudla, Rapoport, Yang.** The quantity \( c(D_2) := \log J_{\infty}(D_1, D_2) \) (with \( D_1 \) fixed) is the \( D_1D_2 \)-th fourier coefficient of a *mock modular form* of weight 3/2.
If $\tau$ is a real quadratic irrationality, then $j(\tau)$ is not defined...!

It is a part of “Kronecker’s jugendtraum” or Hilbert’s twelfth problem, to “make sense” of $j(\tau)$ in this setting.

**Goals of this lecture**: For $\tau_1$ and $\tau_2$ real quadratic,

- construct $J_p(\tau_1, \tau_2) \in H_{12}$ by $p$-adic analytic means;
- relate $\text{ord}_q J_p(\tau_1, \tau_2)$ to the topological intersection of certain real quadratic geodesics on Shimura curves.
- interpret the generating series for $\log_p J_p(\tau_1, \tau_2)$ in terms of certain “$p$-adic mock modular forms”. 
The Drinfeld $p$-adic upper half plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ offers a tempting framework for “real multiplication theory”, since, unlike $\mathcal{H}$, it contains an abundance of real quadratic irrationalities.

**Definition**

A point on $\tau \in \mathcal{H}_p$ is called a *real multiplication (RM) point* if it belongs to $\mathcal{H}_p \cap K$ for some real quadratic field $K$.

**Hope**: A $p$-adic analogue of $j$ leads to singular moduli for real quadratic $\tau \in \mathcal{H}_p$.

**Question**: What is this $p$-adic analogue?
Rigid meromorphic functions on $\mathcal{H}_p$

**Classical setting:** Meromorphic functions on $\text{SL}_2(\mathbb{Z})\backslash \mathcal{H}$.

**The $p$-adic setting:** A rigid meromorphic function is a ratio of rigid analytic functions.

It is natural to consider rigid meromorphic functions with good transformation properties under $\text{SL}_2(\mathbb{Z})$.

In fact it turns out to be appropriate to work with an even larger group of symmetries: the $p$-modular group

$$\Gamma := \text{SL}_2(\mathbb{Z}[1/p]).$$
The action of $\Gamma$, or even of $\text{SL}_2(\mathbb{Z})$, on $\mathcal{H}_p$ is not discrete in the $p$-adic topology. The subgroup of translations $z \mapsto z + n$, with $n \in \mathbb{Z}$, already has non-discrete orbits!

Let $\mathcal{M} :=$ the space of rigid meromorphic functions on $\mathcal{H}_p$, endowed with the translation action of $\Gamma$:

$$f|_\gamma = f\left(\frac{az + b}{cz + d}\right).$$

There are no non-constant $\text{SL}_2(\mathbb{Z})$ or $\Gamma$-invariant elements in $\mathcal{M}$:

$$H^0(\Gamma, \mathcal{M}) = \mathbb{C}_p.$$
Rigid meromorphic cocycles

Let $\mathcal{M}^\times :=$ the multiplicative group of non-zero elements of $\mathcal{M}$. Since $H^0(\Gamma, \mathcal{M}^\times) = \mathbb{C}_p^\times$, consider its higher cohomology instead!

**Definition**

A *rigid meromorphic cocycle* is a class in $H^1(\Gamma, \mathcal{M}^\times)$.

It is said to be *parabolic* if its restrictions to the parabolic subgroups of $\Gamma$ are trivial.

Elementary but key observation: rigid meromorphic cocycles can be meaningfully *evaluated* at RM points.
Evaluating a modular cocycle at an RM point

An element $\tau \in \mathcal{H}_p$ is an RM point if and only if

$$\text{Stab}_\Gamma(\tau) = \langle \pm \gamma_\tau \rangle$$

is an infinite group of rank one.

**Definition**

If $J \in H^1(\Gamma, M^\times)$ is a rigid meromorphic cocycle, and $\tau \in \mathcal{H}_p$ is an RM point, then the value of $J$ at $\tau$ is

$$J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p \cup \{\infty\}.$$

The quantity $J[\tau]$ is a well-defined numerical invariant, independent of the cocycle representing the class of $J$, and

$$J[\gamma \tau] = J[\tau], \quad \text{for all } \gamma \in \Gamma.$$
Rigid meromorphic cocycles and RM points

Let $S$ be the standard matrix of order 2 in $\text{SL}_2(\mathbb{Z})$.

**Theorem (Jan Vonk, D)**

If $J$ is a rigid meromorphic cocycle, then $j := J(S) \in \mathcal{M}^\times$ has its poles concentrated in finitely many $\Gamma$-orbits of RM points.

$H_\tau :=$ ring class field attached to the prime-to-$p$-part of $\text{disc}(\tau)$.

**Definition**

The *field of definition of $J$, denoted $H_J$, is the compositum of $H_\tau$ as $\tau$ ranges over the poles of $j(z)$.*
The main conjecture of real multiplication

The main assertion of complex multiplication:

**Theorem (Kronecker, ...)**

Let $J$ be a meromorphic modular function on $\text{SL}_2(\mathbb{Z})\backslash \mathcal{H}$ with Fourier coefficients in a field $H_J$. For all imaginary quadratic $\tau \in \mathcal{H}$, the value $J(\tau)$ belongs to the compositum of $H_J$ and $H_\tau$.

**Conjecture (Jan Vonk, D)**

Let $J$ be a rigid meromorphic cocycle on $\text{SL}_2(\mathbb{Z}[1/p])\backslash \mathcal{H}_p$, and let $H_J$ denote its field of definition. For all real quadratic $\tau \in \mathcal{H}_p$, the value $J[\tau]$ belongs to the compositum of $H_J$ and $H_\tau$. 
Example of rigid meromorphic cocycles

For real quadratic \( \tau \in \mathcal{H}_p \), the orbit \( \Gamma_\tau \) is dense in \( \mathcal{H}_p \).

The set \( \Sigma_\tau := \{ w \in \Gamma_\tau \text{ such that } ww' < 0 \} \) is discrete.

**Theorem (Vonk, D)**

Let \( p = 2, 3, 5, 7, 11, 17, 19, 23, 29, 31, 41, 47, 59, \text{ or } 71. \) (i.e., \( p \) divides the cardinality of the Monster sporadic group!) For each real quadratic \( \tau \in \mathcal{H}_p \), there is a unique rigid meromorphic cocycle \( J^+_\tau \) for which \( j^+_\tau := J^+_\tau(S) \) is given by

\[
j^+_\tau(z) \sim \prod_{w \in \Sigma_\tau, |w|_p \leq 1} \left( \frac{z - w}{z - pw} \right)^{\text{sgn}(w)} \times \prod_{w \in \Sigma_\tau, |w|_p > 1} \left( \frac{z/w - 1}{z/pw - 1} \right)^{\text{sgn}(w)}.
\]
Rigid meromorphic cocycles are amenable to explicit numerical calculations on the computer, for the following reasons:

• The rigid meromorphic cocycle $J$ is completely determined by a single rigid analytic function $j := J(S) \in \mathcal{O}_{\mathcal{H}_p}$.

• The value $J[\tau]$ can be expressed as a product of values of the form $j(w)$ where $w$ belongs to the “standard affinoid” $A \subset \mathcal{H}_p$, namely, the complement of the $p + 1 \mod p$ residue discs centered at the points in $\mathbb{P}_1(\mathbb{F}_p)$.

• The image of $j(z)$ in the Tate algebra $\mathcal{O}_{A}$ can be computed with an accuracy of $p^{-M}$ in time that is polynomial in $M$. 
An example

Let \( \varphi = \frac{-1 + \sqrt{5}}{2} \) be the golden ratio.

The \( p \)-adic \( J^+_\varphi \) for \( p = 2, 3, 7, 13, 17, 23, \text{ or } 47 \), is the “simplest instance” of a rigid meromorphic cocycle.

The RM point \( \tau = \sqrt{223} \) of discriminant 223 has class number 6, and \( J^+_\varphi[\sqrt{223}] \) appears to satisfy:

\[
\begin{align*}
p = 7. & \quad 282525425x^6 + 27867770x^5 + 414793887x^4 - 128906260x^3 + 414793887x^2 + 27867770x + 282525425, \\
p = 13. & \quad 464800x^6 + 1275520x^5 + 1614802x^4 + 1596283x^3 + 1614802x^2 + 1275520x + 464800, \\
p = 47. & \quad 4x^6 + 4x^5 + x^4 - 2x^3 + x^2 + 4x + 4.
\end{align*}
\]
Rigid meromorphic cocycles are analogous to *rational modular cocycles*: elements $\Phi \in H^1(\text{SL}_2(\mathbb{Z}), \mathcal{R}^\times)$, where $\mathcal{R}^\times$ is the multiplicative group of rational functions on $\mathbb{P}_1$.

- These objects were studied and classified by Marvin Knopp, Avner Ash, Youngju Choie and Don Zagier.

- Bill Duke, Ozlem Imamoglu, Arpad Toth: the RM values of rational modular cocycles are related to the topological linking numbers of real quadratic geodesics on $\text{SL}_2(\mathbb{Z})\backslash\text{SL}_2(\mathbb{R})$. 
Guided by the Knopp-Choie-Zagier classification, we have:

**Theorem (Jan Vonk, D)**

*For any RM point $\tau \in \mathcal{H}_p$, there is a unique $J_\tau \in H^1(\Gamma, \mathcal{M}^\times / \mathbb{C}_p^\times)$ whose poles are concentrated on $\Gamma_\tau$. Every rigid meromorphic cocycle is a product of powers of finitely many of these $J_\tau$, modulo scalars.*

The definition of $J_\tau$ is very similar to that of $J_\tau^+$. 

**Remark:** $H^1_{\text{par}}(\Gamma, \mathbb{C}_p^\times)$ is trivial, so a rigid meromorphic cocycle is determined by its image in $H^1(\Gamma, \mathcal{M}^\times / \mathbb{C}_p^\times)$. 
The work of Duke, Imamoglu and Toth on linking number of modular geodesics immediately suggests the following definition:

**Definition:** The quantity \( J_p(\tau_1, \tau_2) := J_{\tau_1}[\tau_2] \in H_{12} \) is called the \( p \)-adic intersection number of \( \tau_1 \) and \( \tau_2 \).

**Conjecture (Jan Vonk, D)**

The quantity \( J_p(\tau_1, \tau_2) \) behaves in many key respects like the classical \( J_{\infty}(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2) \) of Gross-Zagier.
A few values of $J_p(\sqrt{2}, \tau)$ with $\tau \in \mathbb{Z}[\sqrt{2}]$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$p = 3$</th>
<th>$p = 5$</th>
<th>$p = 13$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\sqrt{2}$</td>
<td>$\frac{7+24\sqrt{-1}}{2.5^2}$</td>
<td>$\frac{-7+4\sqrt{-2}}{3^2}$</td>
<td>1</td>
</tr>
<tr>
<td>$4\sqrt{2}$</td>
<td>$\frac{-7+24\sqrt{-1}}{2.5^2}$</td>
<td>$\frac{-7+4\sqrt{-2}}{3^2}$</td>
<td>1</td>
</tr>
<tr>
<td>$7\sqrt{2}$</td>
<td>$\frac{-97247+24675\sqrt{-7}}{2^4.11^4}$</td>
<td>$\frac{-2719+5763\sqrt{-7}}{2^7.11^2}$</td>
<td>$\frac{31+3\sqrt{-7}}{2^5}$</td>
</tr>
<tr>
<td>$8\sqrt{2}$</td>
<td>$\frac{2047+3696\sqrt{-1}}{5^2.13^2}$</td>
<td>$\frac{511+680\sqrt{-2}}{3^2.11^2}$</td>
<td>$\frac{7+4\sqrt{-2}}{3^2}$</td>
</tr>
<tr>
<td>$11\sqrt{2}$</td>
<td>$\frac{-17005265613+1565252064\sqrt{-22}}{13^2.19^4.29^2}$</td>
<td>$\frac{28463+504\sqrt{-22}}{13^4}$</td>
<td>$\frac{-8071+2363\sqrt{-11}}{2.3^2.5^4}$</td>
</tr>
<tr>
<td>$16\sqrt{2}$</td>
<td>$\frac{985306661831273376-3358763261719606193\sqrt{-1}}{5^6.13^2.29^4.37^4}$</td>
<td>$\frac{651578431+788458960\sqrt{-2}}{3^6.11^6}$</td>
<td>$\frac{-7+4\sqrt{-2}}{3^2}$</td>
</tr>
</tbody>
</table>
Gross-Zagier factorisations

\( J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2) \in H_{12} = H_1 H_2. \)

Fix embeddings of \( H_{12} \) into \( \mathbb{C} \) and into \( \overline{\mathbb{Q}}_q \), for each \( q \).

We can then talk about \( \text{ord}_q J_\infty(\tau_1, \tau_2) \).

Gross and Zagier gave an algebraic formula for this quantity, involving the definite quaternion algebra \( B_{q\infty} \) satisfying:

- \( B_{q\infty} \otimes \mathbb{R} \simeq H \), where \( H = \) Hamilton quaternions;
- \( B_{q\infty} \otimes \mathbb{Q}_q \simeq H_q \), the unique division algebra of rank 4 over \( \mathbb{Q}_q \);
- \( B_{q\infty} \otimes \mathbb{Q}_l \simeq M_2(\mathbb{Q}_l) \), for all \( l \neq \infty, q \).
Quaternionic embeddings

A CM point $\tau \in \mathcal{H}$ of discriminant $D < 0$ corresponds to an embedding of the order $\mathcal{O}$ into $M_2(\mathbb{Z})$, the maximal order in the split quaternion algebra $M_2(\mathbb{Q})$.

**Definition:** An optimal embedding of $\mathcal{O}$ into $B_{q\infty}$ is a pair $(\varphi, R)$ where $R$ is a maximal order in $B_{q\infty}$ and $\varphi: K \to B_{q\infty}$ satisfies $\varphi(K) \cap R = \varphi(\mathcal{O})$.

The group $B_{q\infty}^\times$ acts on $\text{Emb}(\mathcal{O}, B_{q\infty})$ by conjugation:

$$b \ast (\varphi, R) = (b\varphi b^{-1}, bRb^{-1}).$$

$$\Sigma(\mathcal{O}, B_{q\infty}) := B_{q\infty}^\times \backslash \text{Emb}(\mathcal{O}, B_{q\infty}).$$

**Key Fact:** Both $\text{SL}_2(\mathbb{Z})\backslash \mathcal{H}^D$ and $\Sigma(\mathcal{O}, B_{q\infty})$ are endowed with simply transitive $G_D$-actions.
Arithmetic intersection multiplicities

• Given $(\varphi_1, R_1) \in \text{Emb}(\mathcal{O}_1, B_{q\infty})$ and $(\varphi_2, R_2) \in \text{Emb}(\mathcal{O}_2, B_{q\infty})$, let $[\varphi_1, \varphi_2]_q = 0$ if $R_1 \neq R_2$, and, if $R_1 = R_2 =: R$,

$$[\varphi_1, \varphi_2]_q := \text{Max}_t \text{ such that } \varphi_1(\mathcal{O}_1) = \varphi_2(\mathcal{O}_2) \text{ in } R/q^{t-1}R.$$ 

• Given $(\varphi_1, R_1) \in \Sigma(\mathcal{O}_1, B_{q\infty})$ and $(\varphi_2, R_2) \in \Sigma(\mathcal{O}_2, B_{q\infty})$, set

$$(\varphi_1, \varphi_2)_q := \sum_{b \in B_{q\infty}^\times} [b\varphi_1 b^{-1}, \varphi_2]_q.$$
The Gross-Zagier factorisation

**Theorem (Gross-Zagier)**

Let $q 
mid D_1 D_2$ be a prime. If $D_1$ or $D_2$ is a square modulo $q$, then $\text{ord}_q J_\infty(\tau_1, \tau_2) = 0$. Otherwise, there exists bijections

$$\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^{D_1} \leftrightarrow \Sigma(\mathcal{O}_{D_1}, B_{q\infty}), \quad \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^{D_2} \leftrightarrow \Sigma(\mathcal{O}_{D_2}, B_{q\infty}),$$

compatible with the $G_{D_1}$ and $G_{D_2}$-actions, for which

$$\text{ord}_q J_\infty(\tau_1, \tau_2) = (\varphi_1, \varphi_2)_q,$$

for all $\tau_1 \in \mathcal{H}^{D_1}$ and $\tau_2 \in \mathcal{H}^{D_2}$, associated to $\varphi_1$ and $\varphi_2$ respectively.
We now consider the factorisation of \( J_p(\tau_1, \tau_2) \in H_{12} = H_1 H_2 \).

Fix embeddings of \( H_{12} \) into \( \mathbb{C} \) and into \( \bar{\mathbb{Q}}_q \), for each \( q \).

We can then talk about \( \text{ord}_q J_p(\tau_1, \tau_2) \).

Our conjectural formula for this quantity, involves... the **indefinite** quaternion algebra \( B_{qp} \) ramified at \( q \) and \( p \):

- \( B_{qp} \otimes \mathbb{Q}_q \simeq H_q \), \( B_{qp} \otimes \mathbb{Q}_p \simeq H_p \), the unique division algebra of rank 4 over \( \mathbb{Q}_q \) and \( \mathbb{Q}_p \);

- \( B_{qp} \otimes \mathbb{Q}_\ell \simeq M_2(\mathbb{Q}_\ell) \), for all \( \ell \neq p, q \).

- \( B_{qp} \otimes \mathbb{R} \simeq M_2(\mathbb{R}) \);
Because $B_{qp}$ in indefinite, it has a unique maximal order $R$, up to conjugation.

The group $\Gamma_{pq} = R_1^\times \subset SL_2(\mathbb{R})$ acts discretely and co-compactly on $\mathcal{H}$;

The Riemann surface $\Gamma_{pq}\backslash \mathcal{H}$ is called the Shimura curve attached to the pair $(p, q)$.

Given embeddings $\varphi_1 \in \text{Emb}(\mathcal{O}_1, R)$ and $\varphi_2 \in \text{Emb}(\mathcal{O}_2, R)$, let $\gamma_1$ and $\gamma_2$ be the hyperbolic geodesics on $\mathcal{H}$ joining the fixed points for $\varphi_1(\mathcal{O}_1^\times)$ and $\varphi_2(\mathcal{O}_2^\times)$ respectively.

The geodesics $\gamma_1$ and $\gamma_2$ map to closed geodesics $\bar{\gamma}_1$ and $\bar{\gamma}_2$ on the Shimura curve $\Gamma_{pq}\backslash \mathcal{H}$.
Topological intersections

$[\gamma_1, \gamma_2]_\infty := \text{signed intersection of } \gamma_1 \text{ and } \gamma_2.$

**Fact.** The topological intersection multiplicity of $\bar{\gamma}_1$ and $\bar{\gamma}_2$ on the Shimura curve $\Gamma_{pq}\backslash \mathcal{H}$ is

$$(\bar{\gamma}_1, \bar{\gamma}_2)_\infty := \sum_{b \in \mathcal{O}_2^\times \setminus \Gamma_{pq}/\mathcal{O}_1^\times} [b\gamma_1 b^{-1}, \gamma_2]_\infty.$$ 

**Definition.** The $q$-weighted intersection number of $\varphi_1$ and $\varphi_2$ is

$$(\varphi_1, \varphi_2)_q\infty := \sum_{b \in \mathcal{O}_2^\times \setminus \Gamma_{pq}/\mathcal{O}_1^\times} [b\varphi_1 b^{-1}, \varphi_2]_q \cdot [b\gamma_1 b^{-1}, \gamma_2]_\infty.$$
A Gross-Zagier-style factorisation

Conjecture (Jan Vonk, D)

Let $q \nmid D_1 D_2$ be a prime. If $D_1$ or $D_2$ is a square modulo $q$, then $\text{ord}_q J_p(\tau_1, \tau_2) = 0$. Otherwise, there exists bijections

$$\Gamma \backslash \mathcal{H}_p^{D_1} \leftrightarrow \Sigma(\mathcal{O}_{D_1}, R), \quad \Gamma \backslash \mathcal{H}_p^{D_2} \leftrightarrow \Sigma(\mathcal{O}_{D_2}, R),$$

which are compatible with the $G_{D_1}$ and $G_{D_2}$-actions, and for which

$$\text{ord}_q J_p(\tau_1, \tau_2) = (\varphi_1, \varphi_2)_{q\infty},$$

for all $\tau_1 \in \mathcal{H}_p^{D_1}$ and $\tau_2 \in \mathcal{H}_p^{D_2}$, associated to $\varphi_1$ and $\varphi_2$ respectively.
An example

James Rickards has developed and implemented efficient algorithms for computing the $q$-weighted topological intersection numbers of real quadratic geodesics on Shimura curves.
An example: $D_1 = 13$, $D_2 = 285 = 3 \cdot 5 \cdot 19$, $p = 2$

**Vonk, D**: $J_2(\tau_1, \tau_2)$ satisfies (to 800 digits of 2-adic precision)

\[
77360972841758936947502973998239x^4 + 140181070438890831721314135099803x^3 \\
+ 209895619549791255199413489899292x^2 + 140181070438890831721314135099803x \\
+ 77360972841758936947502973998239,
\]

**James Rickards**: $e_{q2} := \frac{1}{2} \sum_{\tau_1, \tau_2} |(\varphi_{\tau_1}, \varphi_{\tau_2})_{q\infty}|$ on $\Gamma_{2q}\backslash \mathcal{H}$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>7</th>
<th>19</th>
<th>31</th>
<th>73</th>
<th>109</th>
<th>151</th>
<th>163</th>
<th>397</th>
<th>457</th>
<th>463</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{q2}$</td>
<td>7</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**But**: $77360972841758936947502973998239 = 7^7 \cdot 19^2 \cdot 31^2 \cdot 73 \cdot 109^2 \cdot 151^2 \cdot 163 \cdot 397 \cdot 457 \cdot 463$. 
Let \( q \equiv 3 \pmod{4} \) be a prime, and for all negative \( D \),

\[
J_\infty(-q, D) := \prod_{\text{disc}(\tau_1) = -q, \text{disc}(\tau_2) = D} J_\infty(\tau_1, \tau_2) \in \mathbb{Z}.
\]

**Gross-Zagier, Kudla-Rapoport-Yang:** The quantity 
\[ c(D) := \log J_\infty(-q, -D) \] for \( D > 0 \) is the \( D \)-th fourier coefficient of a non-holomorphic modular form of weight 3/2.

This assertion is a very special case of the “Kudla program”, predicting that quantities like \( c(D) \), which describe the arithmetic intersections of natural cycles on Shimura varieties, can be packaged into a modular generating series.
The incoherent Eisenstein series of Kudla-Rapoport-Yang

Let \( \chi_q : (\mathbb{Z}/q\mathbb{Z})^\times \to \pm 1 \) be the odd quadratic Dirichlet character.

Non-homomorphic Eisenstein series:

\[
E_-(\tau, s) = y^{s/2} \sum_{(c,d)} (c\tau + d)^{-1} |c\tau + d|^{-s} \Phi_q^{-}(c, d).
\]

Functional equation: \( E_-(\tau, -s) \sim -E_-(\tau, s) \).

Hence \( E_-(\tau, 0) = 0 \).

**Definition.** The incoherent Eisenstein series of Kudla-Rapoport-Yang is the derivative

\[
\Phi_{KRY} := \frac{d}{ds}(E_-(\tau, s))_{s=0}.
\]

It is a non-holomorphic modular form of weight one.
The quantity $c(D) := \log J_\infty(-q, -D)$ is essentially the $D$th Fourier coefficient of $\Phi_{KRY}(4\tau) \times \theta(\tau)$, where $\theta(q)$ is the standard unary theta series of weight $1/2$.

This theorem has been extended to the setting where weight one theta-series are replaced by a weight one cusp form $g$, by Bill Duke, Yingkun Li, Stephan Ehlen, Maryna Viazovska, and Pierre Charollois+Yingkun Li.

The role of the incoherent Eisenstein series of weight one of KRY is played by a mock modular form of weight one having $g$ as its shadow.
Twisted norms of real quadratic singular moduli

Now let $\psi : G_q \rightarrow L^\times$ be any class character, $q = 1 + 4m$.

The set $\Gamma \backslash \mathcal{H}_p^{\text{disc}=q}$ is endowed with a simple transitive $G_q$-action, and can thus be identified with $G_q$.

For all positive $D$, let

$$J_p(\psi, D) := \prod_{\substack{\text{disc}(\tau_1) = q \\ \text{disc}(\tau_2) = D}} J_p(\tau_1, \tau_2)^{\psi^{-1}(\tau_1)} \in (H_q^\times \otimes L)^\psi.$$

Conjecture (Jan Vonk, D)

The quantity $c_\psi(D) := \log_p J_p(\psi, D)$ is the $D$th fourier coefficient of a “$p$-adic mock modular form” of weight $3/2$. 

---

$\Box$
A $p$-adic Kudla-Rapoport-Yang theorem

**Theorem (Alan Lauder, Victor Rotger, D)**

There exists a “$p$-adic mock modular form” $\Phi_\psi$ of weight one whose Fourier coefficients are the $p$-adic logarithms of elements of $(H_q^\times \otimes L)_\psi$. It exhibits many of the same properties as $\Phi_{KRY}$ and of the forms arising in Duke-Li, Ehlen, Viazovska, Charollois-Li...

The modular form $\Phi_\psi$ is simply the derivative, with respect to the weight, of a Hida family of modular forms specialising to $\theta_{\psi_o}$ in weight one, where $\psi_o/\psi'_o = \psi$.

The proof of the theorem is very different, and **substantially simpler** from the approaches of Kudla-Rapoport-Yang, Duke-Li, Ehlen, Viazovska, Charollois-Li used in the Archimedean setting. It rests crucially on the deformation theory of modular forms and of $p$-adic Galois representations.
A more tractable conjecture?

**Conjecture (Jan Vonk, D)**

The quantity $c_{\psi}(D) := \log_p J_p(\psi, D)$ is essentially the $D$th Fourier coefficient of $\Phi_\psi(q^4) \times \theta(q)$, where $\theta(q)$ is the standard unary theta series of weight $1/2$.

This conjecture suggests a possible road map for proving the algebraicity of “real quadratic singular moduli”...
The RM values of rigid meromorphic multiplicative cocycles lead to *conjectural analogues* of singular moduli, with applications to

- explicit class field theory for real quadratic fields;
- Gross-Zagier style factorisation formulae;
- suggesting test cases for an eventual “$p$-adic Kudla program”.

The experiments reveal a promising connection between the $p$-adic Kudla program and Hilbert’s twelfth problem for real quadratic fields.

We are still *very far* from understanding this “real multiplication theory” as well as its classical counterpart!
Thank you for your attention!