What features contribute to stability?

When force chains branch, they do so in either a densely packed or loosely packed fashion, with a gap between the branches.

In the former case, the stability to normal forces on the branch particles increased at the branch point, and at the latter it is decreased.
Define a (topological) statistic to capture this.

Consider the clique complex of the graph.
- “Dense” branches correspond to 2-cliques in the force chain graph.
- “Loose” branches either result in leaves, or internal loops, so $\beta_1(G/L)$ (roughly) counts “loose” branches.
- The percentage of branches which are loosely packed is given by:

$$\bar{\beta}_1(G/L) = \beta_1(G/L) / \text{rank}(\ker(\delta_1))$$
Define a (topological) statistic to capture this.

The systemic kernel normalized first Betti number of a force chain network with $n$ communities $\{c\}$ is

$$\bar{\beta}_{1,s} = \sum_c \bar{\beta}_1(c/L_c)s_c / n \cdot s_{\text{max}}$$

where $s_c$ is the size of community $c$. 
This is distinct from the gap factor...
...and stratifies pressure at $\gamma \sim 1$. 
We plan to extend this to a dynamic statistic.

Q is a multilayer modularity function which we again optimize using gradient descent to obtain a sequence of community structures \( \{c_i\} \).
We plan to extend this to a dynamic statistic.

Given partitions (force chains) \{c\} and \{c'\} for networks at adjacent parameter choices, we can construct the mutual refinement \{c\} \cap \{c'\}, which admits canonical maps to each partition.

These induce maps

\[ H_i(\{c\} \cap \{c'\}) \]

\[ H_i(\{c\}) \quad \text{ and } \quad H_i(\{c'\}) \]
We plan to extend this to a dynamic statistic.

Using the multilayer community structure, we obtain a “zig-zag” diagram:

\[
\begin{align*}
H_i(\{c_1\} \cap \{c_2\}) & \quad H_i(\{c_2\} \cap \{c_3\}) & \quad \ldots \\
H_i(\{c_1\}) & \quad H_i(\{c_2\}) & \quad H_i(\{c_3\})
\end{align*}
\]

Standard algebraic methods (pushouts) will now allow us to extract “persistence” of cycles: when they form and disappear, and how their size/shape change as the parameter varies, detecting change in structure under varying pressure/shear, etc.