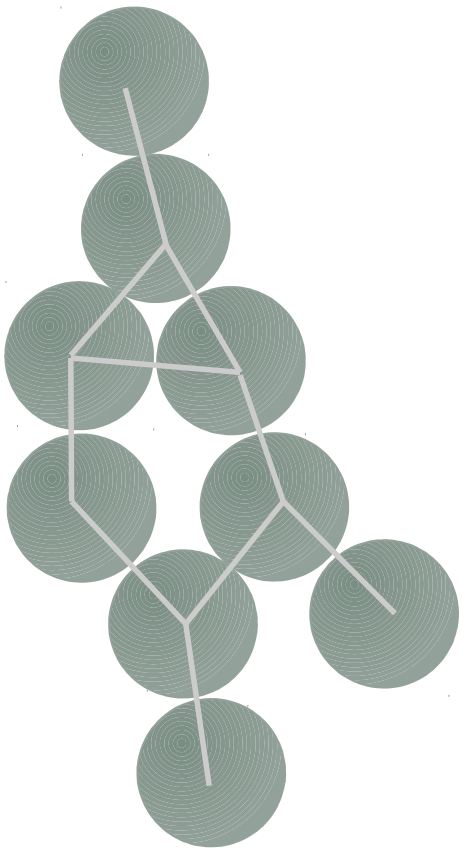
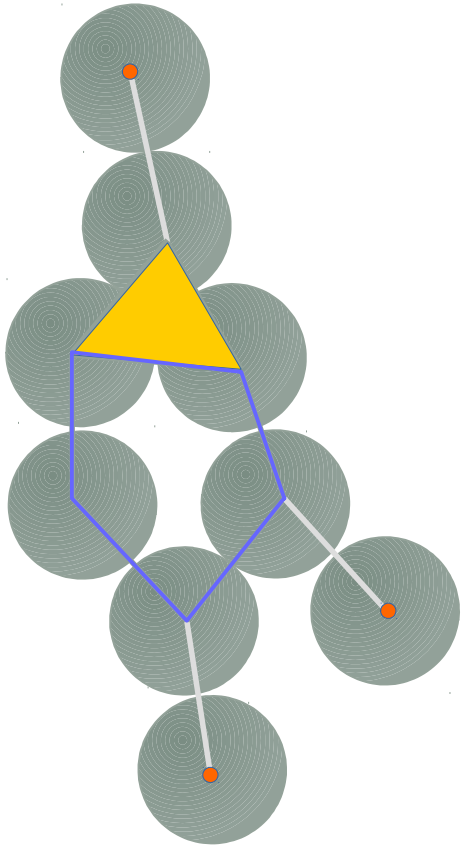
What features contribute to stability?



When force chains branch, they do so in either a densely packed or loosely packed fashion, with a gap between the branches.

In the former case, the stability to normal forces on the branch particles increased at the branch point, and at the latter it is decreased.

Define a (topological) statistic to capture this.

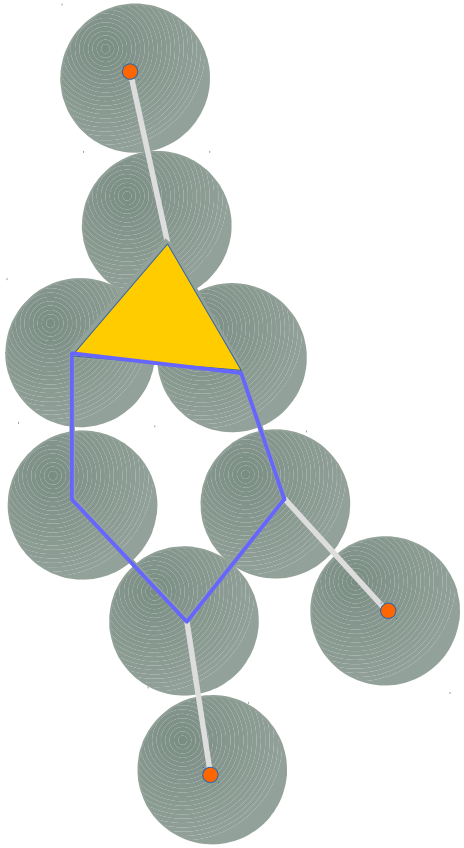


Consider the clique complex of the graph.

- "Dense" branches correspond to 2-cliques in the force chain graph.
- "Loose" branches either result in leaves, or internal loops, so $\beta_1(G/L)$ (roughly) counts "loose" branches.
- The percentage of branches which are loosely packed is given by:

$$\bar{\beta}_1(G/L) = \beta_1(G/L) / \mathrm{rank}(\ker(\delta_1))$$

Define a (topological) statistic to capture this.

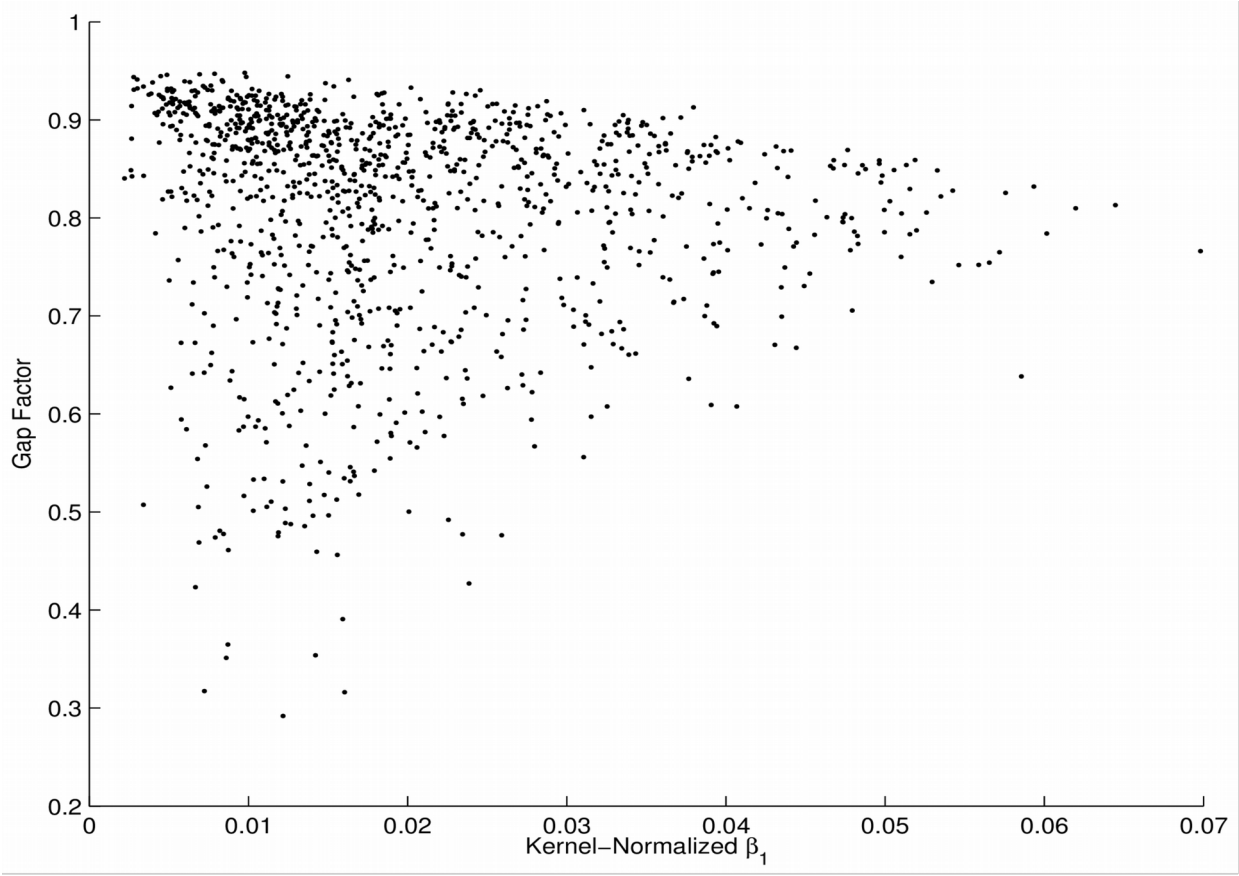


The *systemic kernel normalized first Betti number* of of a force chain network with *n* communities {c} is

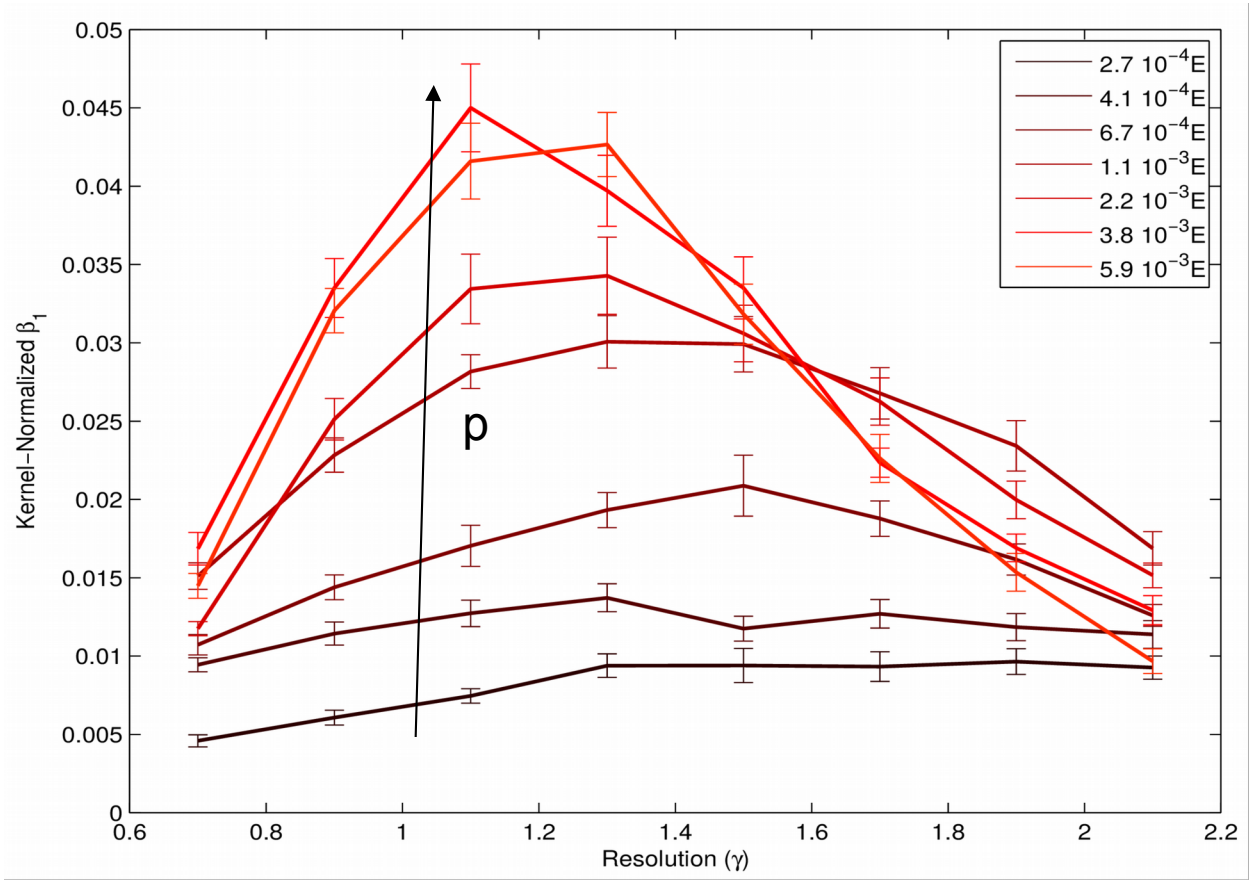
$$\overline{\beta}_{1,s} = \Sigma_c \, \overline{\beta}_1(c/L_c)s_c \, / \, n \cdot s_{max}$$

where s_c is the size of community c.

This is distinct from the gap factor...

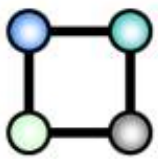


...and stratifies pressure at $\gamma \sim 1$.



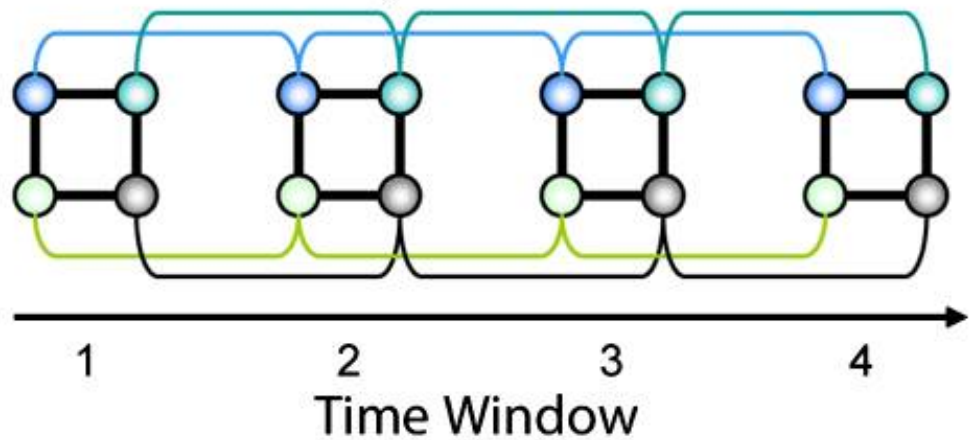
We plan to extend this to a dynamic statistic.

Single-Layer Framework



Network From One Time Window

Multilayer Framework



$$Q = \frac{1}{2\mu} \sum_{ijlr} \{ (A_{ijl} - \gamma_l P_{ijl}) \delta_{lr} + \delta_{ij} \omega_{jlr} \} \delta(g_{il}, g_{jr})$$

Q is a *multilayer modularity function* which we again optimize using gradient descent to obtain a sequence of community structures $\{c_l\}$

We plan to extend this to a dynamic statistic.

Given partitions (force chains) {c} and {c'} for networks at adjacent parameter choices, we can construct the mutual refinement {c}∩{c'}, which admits canonical maps to each partition.

These induce maps

$$\begin{tikzcd} & H_i(\{c\}\cap\{c'\}) \arrow[dl] \arrow[dr] & \\ H_i(\{c\}) & & H_i(\{c'\}) \end{tikzcd}$$

We plan to extend this to a dynamic statistic.

Using the multilayer community structure, we obtain a “zig-zag” diagram:

$$\begin{tikzcd}
 & H_i(\{c_1\}\cap\{c_2\}) \arrow[dl] \arrow[dr] & & H_i(\{c_2\}\cap\{c_3\}) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \\
H_i(\{c_1\}) & & H_i(\{c_2\}) & & H_i(\{c_3\}) &
\end{tikzcd}$$

Standard algebraic methods (pushouts) will now allow us to extract “persistence” of cycles: when they form and disappear, and how their size/shape change as the parameter varies, detecting change in structure under varying pressure/shear, etc.