Problems in local Galois deformation theory

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Modularity

**Theorem (Wiles, Taylor-Wiles, BCDT):** Any elliptic curve $E/\mathbb{Q}$ is modular.

An elliptic curve is **modular** if $\rho_E : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{Q}_p)$ is isomorphic to $\rho_f$ for some modular form $f$. 
Modularity lifting

For any $p$-adic representations $\rho$, let $\bar{\rho}$ denote the (semi-simplified) reduction mod $p$.

**Modularity lifting prototype:** If $\rho$ is a $p$-adic representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfying conditions $X$, $Y$, and $Z$, then $\bar{\rho}$ modular implies $\rho$ is modular.
Local conditions

The most common conditions are conditions on the restriction of $\rho$ to the decomposition groups $\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ for each prime $\ell$ and the most subtle of these occur when $\ell = p$.

Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $\Gamma_K$ be the absolute Galois group of $K$. Fix

$$\bar{\rho} : \Gamma_K \to \text{GL}_n(\mathbb{F}_p).$$

There is a universal local deformation space $D_{\bar{\rho}}$. 
Flat deformations

The flat deformation space $D^\text{fl}_\rho$ is the subspace of $D_\rho$ consisting of representations that come from finite flat group schemes over the ring of integers $\mathcal{O}_K$ of $K$.

**Example:** If $E$ is an elliptic curve over $K$ with good reduction, then the Galois representations on the $p^n$-torsion points of $E$ are flat for all $n$. In particular, $\rho_E$ lies in $D^\text{fl}_\rho$. 
Generalizations

- (Higher weight) Crystalline deformation spaces $D_{\bar{\rho}}^{\text{cris},\mu}$ with Hodge type $\mu$ ($D_{\bar{\rho}}^{\text{fl}}$ is essentially the case of Hodge type $\{0, 1\}$.)
- (Higher level) Semi-stable deformation spaces $D_{\bar{\rho}}^{\text{st},\mu}$
Questions

1. What are the connected components of $D^\text{fl}_{\rho}[1/p]$? $D^\text{cris,}\mu_{\rho}[1/p]$? $D^\text{st,}\mu_{\rho}[1/p]$?

2. What is the structure mod $p$ of $D^\text{fl}_{\rho}$? $D^\text{cris,}\mu_{\rho}$? $D^\text{st,}\mu_{\rho}$?
   (Breuil-Mézard conjecture)
Progress

Assume $\bar{\rho}$ is irreducible.

- When $K$ is unramified over $\mathbb{Q}_p$ and $\mu$ is "small" relative to $p$, then $D^{\text{cris},\mu}_\rho$ is smooth and has just one component.
- (Kisin, Imai, Gee, Hellmann) In the case of $GL_2$, they answer Question 1 for $D^{\text{fl}}_\rho[1/p]$ with no restrictions $K$.
- There has also been progress on Question 2 for $GL_2$ (see recent work of Gee and Kisin).
Kisin’s ground-breaking technique was to introduce a resolution

$$X^*_{\bar{\rho}, \mu} \to D^*_{\bar{\rho}, \mu}$$

of the * deformation space which is a moduli space of ”linear algebra” data (using deep results from integral $p$-adic Hodge theory).

In the flat case, one understands the singularities of $X^{\text{fl}, \mu}_{\bar{\rho}}$ using local models of a Shimura varieties. This was essential in answering the connected components question for flat deformation spaces when $K$ is ramified.
G-valued Galois deformations

Let $G$ be a reductive group over $\mathbb{Z}_p$. For any $\bar{\rho} : \Gamma_K \to G(\mathbb{F}_p)$, there is a universal space $D_{\bar{\rho}, G}$ of $G$-valued Galois deformations. There are also crystalline and semi-stable subspaces with specified Hodge type $\mu$.

**Theorem(-):** There exists a projective morphism

$$\Theta : X_{\bar{\rho}, G}^{\text{cris}, \mu} \to D_{\bar{\rho}, G}^{\text{cris}, \mu}$$

which is an isomorphism with $p$ inverted. Furthermore, if $\mu$ is sufficiently “small,” then the local structure of $X_{\bar{\rho}, G}^{\text{cris}, \mu}$ is equivalent to that of a local model for the group $\text{Res}_{K/\mathbb{Q}_p} G$. 
Conclusions

- The Theorem on the previous slide is a first step toward answering the connected components question for $G$-valued "flat" deformation spaces.
- One would like to understand the structure of $X_{\text{cris},\mu}^{\rho,G}$ in the higher weight situation ($\mu$ large).
- One hopes results on the connected components question will lead to better modularity lifting theorems.