Problems in local Galois deformation theory

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Modularity

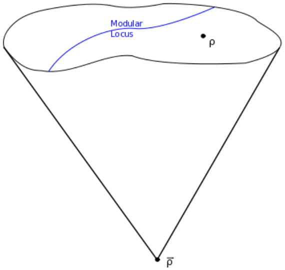
Theorem (Wiles, Taylor-Wiles, BCDT): Any elliptic curve $E/\mathbb{Q}$ is modular.

An elliptic curve is **modular** if $\rho_E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p)$ is isomorphic to ρ_f for some modular form f.

Modularity lifting

For any p-adic representations ρ, let $\overline{\rho}$ denote the (semi-simplified) reduction mod p.

Modularity lifting prototype: If ρ is a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfying conditions X, Y, and Z, then $\overline{\rho}$ modular implies ρ is modular.



Local conditions

The most common conditions are conditions on the restriction of ρ to the decomposition groups $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ for each prime ℓ and the most subtle of these occur when $\ell = p$.

Let K be a finite extension of $\mathbb{Q}_p$ and let Γ_K be the absolute Galois group of K. Fix

$$\overline{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{F}_p).$$

There is a universal local deformation space $D_{\overline{\rho}}$.

Flat deformations

The flat deformation space $D_{\overline{\rho}}^{\text{fl}}$ is the subspace of $D_{\overline{\rho}}$ consisting of representations that come from finite flat group schemes over the ring of integers $\mathcal{O}_K$ of K.

Example: If E is an elliptic curve over K with good reduction, then the Galois representations on the p^n-torsion points of E are flat for all n. In particular, ρ_E lies in $D_{\overline{\rho}}^{\text{fl}}$.

Generalizations

- (Higher weight) Crystalline deformation spaces $D_{\overline{\rho}}^{\mathrm{cris},\mu}$ with Hodge type μ ($D_{\overline{\rho}}^{\mathrm{fl}}$ is essentially the case of Hodge type $\{0,1\}$.)
- (Higher level) Semi-stable deformation spaces $D_{\overline{\rho}}^{\mathrm{st},\mu}$

Questions

1. What are the connected components of $D_{\overline{\rho}}^{\mathrm{fl}}[1/p]$? $D_{\overline{\rho}}^{\mathrm{cris},\mu}[1/p]$? $D_{\overline{\rho}}^{\mathrm{st},\mu}[1/p]$?
2. What is the structure mod p of $D_{\overline{\rho}}^{\mathrm{fl}}$? $D_{\overline{\rho}}^{\mathrm{cris},\mu}$? $D_{\overline{\rho}}^{\mathrm{st},\mu}$? (Breuil-Mézard conjecture)

Progress

Assume $\overline{\rho}$ is irreducible.

- When K is unramified over $\mathbb{Q}_p$ and μ is “small” relative to p, then $D_{\overline{\rho}}^{\text{cris},\mu}$ is smooth and has just one component.
- (Kisin, Imai, Gee, Hellmann) In the case of GL_2, they answer Question 1 for $D_{\overline{\rho}}^{\text{fl}}[1/p]$ with no restrictions K.
- There has also been progress on Question 2 for GL_2 (see recent work of Gee and Kisin).

Technique

Kisin's ground-breaking technique was to introduce a resolution

$$X_{\overline{\rho}}^{*,\mu} \longrightarrow D_{\overline{\rho}}^{*,\mu}$$

of the $*$ deformation space which is a moduli space of "linear algebra" data (using deep results from integral p-adic Hodge theory).

In the flat case, one understands the singularities of $X_{\overline{\rho}}^{\mathrm{fl},\mu}$ using local models of a Shimura varieties. This was essential in answering the connected components question for flat deformation spaces when K is ramified.

G-valued Galois deformations

Let G be a reductive group over $\mathbb{Z}_p$. For any $\overline{\rho} : \Gamma_K \to G(\mathbb{F}_p)$, there is a universal space $D_{\overline{\rho},G}$ of G-valued Galois deformations. There are also crystalline and semi-stable subspaces with specified Hodge type μ.

Theorem(-): There exists a projective morphism

$$\Theta : X_{\overline{\rho},G}^{\text{cris},\mu} \to D_{\overline{\rho},G}^{\text{cris},\mu}$$

which is an isomorphism with p inverted. Furthermore, if μ is sufficiently "small," then the local structure of $X_{\overline{\rho},G}^{\text{cris},\mu}$ is equivalent to that of a local model for the group $\mathrm{Res}_{K/\mathbb{Q}_p} G$.

Conclusions

- The Theorem on the previous slide is a first step toward answering the connected components question for G-valued "flat" deformation spaces.
- One would like to understand the structure of $X_{\bar{\rho},G}^{\mathrm{cris},\mu}$ in the higher weight situation (μ large).
- One hopes results on the connected components question will lead to better modularity lifting theorems.