Time, Space and Monotone Circuits

Christopher Beck

Sept 29 2014
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For even very simple problems, it is sometimes possible to dramatically reduce the space without increasing time much. Other times this doesn’t appear to be the case.
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Example: “CNF-SAT”. Given boolean formula on $n$ variables in CNF, is it satisfiable?

- Trivial algorithm: Exhaustively enumerate all assignments and check. ($O(2^n)$ time, $O(n)$ space)
- If ETH holds, then there is no $O(2^{(1-\epsilon)n})$ time algorithm, even with exponential space.
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Besides concrete problems, the important meta-algorithm ”Dynamic Programming” always trades space for time.

Basic question: What are the limits of the strategy?
Questions like these have been asked almost since the inception of the field.

Much work in “decision tree” / “branching program” models.
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  - Extension to inapproximability for randomized branching programs. [Beame, Saks, Sun, Vee ’02].
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Much work on “pebble games” on DAGs

- Given a directed acyclic graph, a “pebble” may be placed on any node, all of whose predecessors are pebbled, and removed at any time. The goal is to pebble all nodes, in some order, using few pebbles.
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  - Pebble games model the process of allocating registers to compute the result of a specific circuit.

\*\* (Paul, Tarjan ‘77) DAGs built from expanders become exponentially difficult to pebble with reduced space.
\*\* (Lipton, Tarjan ‘80) Any planar DAG of degree \(O(1)\) with \(n\) vertices can be pebbled with \(n^2/3\) pebbles, and \(n^5/3\) steps.
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- Valiant ['76] shows that any algebraic circuit over a finite field computing a linear transformation whose matrix has the property that, any square submatrix is full rank, has the graph-theoretic property of being a 'superconcentrator'.
A superconcentrator of capacity $n$ is a DAG $G = (V, E)$ with two disjoint sets of vertices $I, O \subseteq V$, $|I| = |O| = n$ such that for all subsets $I' \subseteq I$, $O' \subseteq O$ with $|I'| = |O'|$, there exist $|I'|$ vertex-disjoint paths connecting $I'$ and $O'$. 
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- For (restricted) circuits ($AC^0$, monotone, algebraic...)
  - Valiant ['76] shows that for any algebraic circuit over a finite field (or any) computing a linear transformation, whose matrix has the property that any square submatrix is full rank, has the graph-theoretic property of being a 'superconcentrator'.
  - Savage ['77], Tompa ['81], others use such arguments to show that the Fourier Transform and similar e.g. cannot be computed with space $n^{1-\epsilon}$ without using time $n^{1+\epsilon}$. 
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- (B., Beame, Impagliazzo ’12) Tradeoffs even up to exponential space, with superpolynomial blowups in time.
- This result is different from previous time-space tradeoff results in that it technically extends the previously known (tight) lower bounds for time. It is a purely combinatorial argument and doesn’t reduce to pebbling.
Bottleneck Counting Argument

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- Map *input assignments* to *gates* of the circuit and prove that the probability that any assignment goes to a particular gate is small, e.g. $< \epsilon$. 
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- Map *input assignments* to *gates* of the circuit and prove that the probability that any assignment goes to a particular gate is small, e.g. $< \epsilon$.
- Conclude that there are at least $\epsilon^{-1}$ gates.
Bottleneck Counting Argument

- In Haken’s work this map is ad hoc.
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- In Razborov’s “method of approximations”, it’s based on successive approximations (of low complexity) to the gates of the circuit.
- Janos Simon [’97, et.al ’13] points out the commonalities of these.
In [BBI’12], we derived a time space tradeoff, starting from a tight running time lower bound, which we might sketch as follows:

Consider a short proof, and consider the map from the size lower bound. By varying the parameters to define it, obtain a second map. We have

$$\Pr_{\vec{x}}[f_1(\vec{x}) = g] \leq \epsilon$$

and

$$\Pr_{\vec{x}}[f_2(\vec{x}) = g] \leq \epsilon$$

for all gates $$g$$.

However, now prove also that for any $$g_1, g_2$$ that

$$\Pr_{\vec{x}}[f_1(\vec{x}) = g_1 \land f_2(\vec{x}) = g_2] \leq \epsilon^2$$

If the size of the DAG is $$\approx \epsilon - 1$$, we have a weak form of expansion, and morally it implies a space lower bound.
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Open Questions

- Can we use arguments like this to extend monotone circuit size lower bounds to time-space tradeoffs?
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Thanks!