Solitons & Symmetry

IAS Princeton

Hermann Weyl

Symmetry

Invariant Theory

Physics

Peter Goddard

Magnetic Monopoles

Von Neumann

Computers

Soliton

A solution of certain kind of non-linear PDE that "behaves like a particle"

Includes many important cases that turn up in various branches of physics and interesting mathematics.
SOLITONS IN 1+1 DIMENSIONS

\[ \text{Non-linear} \quad u_t + uu_x + u_{xxt} = 0 \]

Solved by interpreting \( u(x,t) \) as the potential for the linear operator \( L(t) = \frac{d^2}{dx^2} + u \) acting on functions of \( x \) (and evolving in time according to KdV)!

Hidden symmetries associated to Lie group \( SL_2 \)

Extensive theory
Solutions in 3 and 4 dimensions

**Ex dim 4** Instantons: Solutions of self-dual Yang-Mills eqns (exploited by Donaldson on general 4-manifolds)

**dim 3** Magnetic monopoles: Solutions of Bogomolny eqn (closely related to SDYM)

Solved (in $\mathbb{R}^4$ or $\mathbb{R}^3$) by using Penrose twistor* theory

Associated linear operator is Dirac operator (natural potential)

$\Rightarrow$ 1+1 dim theory by dimensional reduction (R. Ward, Mason-Woodhouse)
Aim to discuss magnetic monopoles with Platonic symmetries (why?)

Dirac magnetic monopole field produced by point-source of magnetism

(Static) solution of (linear) Maxwell equations with point singularity

In non-Abelian gauge-theorie 't Hooft & Polyakov found soliton solution (no singularity) looked like Dirac monopole near oo
SIMPLIFIED MODEL BPS
MORE TRACTABLE

DATA $SU(2)$ - CONNECTION (POTENTIAL) $A$
on $\mathbb{R}^3$ : COMPONENTS $A_\mu(x) \in SU(2)$
HIGGS FIELD $\phi : \phi(x) \in SU(2)$

BOGOMOLOVY EQUATION $D_A \phi = * F_A$ - CURVATURE (FIELD)
COVARIANT DERIVATIVE
DUALITY OPERATOR
2 FORMS $\rightarrow$ 1- FORMS

BODY CONDITIONS $F_A \rightarrow 0$ AS $|x| \rightarrow \infty$
$|\phi| \rightarrow 1$

TOPOLOGICAL INVARIANT (INTEGER $k$)

DEGREE OF MAP $\phi_{\infty} : S_3 \rightarrow S_1$

$k = \text{MAGNETIC CHARGE}$

$= \text{"NUMBER" OF BASIC MONOPOLES ("PARTICLES")}$
GAUGE - TRANSFORMATIONS

/ AUTOMORPHISMS OF SU(2) - BUNDLE

SOLUTIONS MODULO AUTOMORPHISMS

GIVE MODULI SPACE $M_k$

(USUAL TO USE BASE-POINT (ORIGIN) AND AUTS $g(x)$ WITH $g(0) = 1$)

RESULTS

$M_k$ IS MANIFOLD OF DIMENSION $4k + 2$ ($k$ POSITIONS + $k$ PHASES)

THEOREM (DONALDESON - JARVIS) $k > 0$

$M_k = \text{SPACE OF HOLONOMORPHIC (RATIONAL) MAPS } f : S^2 \rightarrow S^2$

OF DEGREE $k$

NOTE INVERSE ($f \mapsto A \phi$) NOT EXPLICIT
$\mathbb{R}^1$ \hspace{1cm} $f(z) = z$ \hspace{1cm} "Identity Map"

\Rightarrow \hspace{1cm} Basic 1-Monopole

spherically symmetric \hspace{1cm} about origin

(essentially unique)

Solution explicit \hspace{1cm} simple

$\mathbb{R}^2$

special case $f(z) = z^2$

has axial symmetry

"toroidal" shape

general case $f(z) = z^2 - c$

$|c|$ large splits approx. into

2 basic monopoles far apart

90° scattering under direct collision

$M_\alpha$ has natural riemannian metric

(explicit for $\mathbb{R}^2$) and geodesics

describe slow dynamics of

interacting monopoles
**Jacobian** \( J(\chi) \)

**Branch Points**

Points on \( S^1 \) where \( df = 0 \)

If \( \chi = \chi \)

\( \gamma(\gamma(\gamma)) \) homogeneous

\( \gamma(\gamma(\gamma)) \) polynomial deck

\( J(\chi) = \det \left( \begin{array}{cc} \frac{\partial f}{\partial t_0} & \frac{\partial f}{\partial t_1} \\ \frac{\partial g}{\partial t_0} & \frac{\partial g}{\partial t_1} \end{array} \right) \)

Homog. Polyn. degree \( 2k-2 \)

**Classical Invariant**

If \( \Gamma \subset \text{SO}(3) \) (or \( \Gamma' \subset \text{SU}(2) \))

finite subgroup, can look for

\( \Gamma \)-invariant maps \( f(\Gamma \text{ acting on both sides}) \)

**Question**

Given \( J \) degree \( 2k-2 \) is there one (or more) \( f \) deck with \( J(\chi) \neq J \)?

**Note**

Dimensions of spaces equal

\( f \leftrightarrow P_1 C \emptyset \_ \_ \_ \) dim \( 2k-2 \)

Projective class

\( J : \) dim \( 2k-2 \)
Consider case $R=3$

$$2R-2 = 4$$ $J(4)$ is quartic

Interpret problem geometrically

$f$ corresponds to $CP_2$ in $CP_3$ (cubics up to scale)

Let $X \subset CP_3$ be rational cubic curve representing perfect cubes

$f$

$4$ tangents to $X$

$4$ points on $X$ $\rightarrow$ $4$ tangents

$2$ transversal lines
THEOREM (MFA 1951)

4 points on $x$ have an equianharmonic cross-ratio

(Vertices of a regular tetrahedron)

$\iff$ The 2 transversals coincide

$\Rightarrow$ There is a unique 3-monopole with tetrahedral symmetry !!!

$f(z) = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1}$, $k=3$ tetrahedral

$f(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$, $k=4$ octahedral

$f(z) = \frac{z^7 - 2z^5 - 7z^4 - 1}{z^7 + 2z^5 - 7z^4 + 1}$, $k=7$ icosahedral

$J(f) = 0$ gives mid-points of faces

Points of minimum energy

$\sum$
Skyrmions

Classical Model of Nucleus

Data Map \( g : \mathbb{R}^3 \rightarrow SU(2) = S^3 \)

Boy Cond. \( g(x) \rightarrow 1 \) as \( |x| \rightarrow 0 \)

Topological Invariant Integer \( k \)

Degree \( \tilde{g} : S^3 \rightarrow S^3 \)

Interpreted as Number of Protons/Neutrons

Energy \( E(g) = \int_{\mathbb{R}^3} |d g|^2 + \int_{\mathbb{R}^2} |d^2 g|^2 \)

(Static) Skyrmion is Solution minimizing \( E \) (Given by Euler-Lagrange Equation)

Essentially Unique Solution Expected

Analytical Solutions Not Known

Only Numerical Work Difficult Problem

\[ \sum \]
R = 1 Spherically Symmetric Solution (Solve One Numerically) Gives Basic Skyrmion (Hydrogen Nucleus)

R > 1 R Far Separated Basic Skyrmions "Asymptotic Solution"

But When Close Together?

Analogy ?? with Monopoles Suggests Looking for Symmetric Solutions (Manton)

Remarkable Fact !!!!

\[
\begin{align*}
R &= 2 & \text{Torioidal} \\
R &= 3 & \text{Tetrahedral} \\
R &= 4 & \text{Octahedral} \\
R &= 7 & \text{Icosahedral}
\end{align*}
\]

\[\Rightarrow \text{"Insight" for Other Values of } R \text{ Mysterious!} \]
REFERENCE

TOPOLOGICAL SOLITONS

N. MANTON & P. SUTCLIFFE

CUP 2004