SPECTRAL THEORY OF AUTOMORPHIC FORMS
AND ANALYTIC NUMBER THEORY
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1. INTRODUCTION

I shall speak mostly about the $G_{L_2}$ forms, i.e. functions on the upper half-plane

$H = \text{SL}_2(\mathbb{R})/\mathbb{K} = \{z = x + iy ; x \in \mathbb{R}, y \in \mathbb{R}^+ \}$. 

Tools
- Combinatorial identities of sieve type
- Summation formulas of trace type

Theta series

$$\theta(z) = \sum_{m \in \mathbb{Z}} e(m^2z) , \quad e(z) = e^{2\pi i z}$$

$$\theta(\gamma z) = \psi(\gamma) (cz+d)^\frac{1}{2} \theta(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(4).$$

Poisson's formula

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$
2. SPECTRAL THEORY

\[ \Gamma_0(q) = \{ (a,b) \in SL_2(\mathbb{Z}) ; c \equiv 0 \pmod{q} \} \]

\[ [\Gamma_0(1) : \Gamma_0(q)] = q \prod (1 + \frac{1}{p}) \]

Let \( \Gamma \subset SL_2(\mathbb{R}) \) a group acting discontinuously on \( \mathbb{H} \)

\[ \gamma z = \frac{az + b}{cz + d}, \quad \gamma = (a,b) \]

Adelic setting, Hecke operators for congruence groups.

**Automorphic Functions:**

\[ f : \mathbb{H} \rightarrow \mathbb{C} \]

\[ f(\gamma z) = \psi(\gamma) \left( \frac{cz + d}{|cz + d|} \right)^k f(z) \]

\( f(z) \) - polynomial growth at cusps

If \( k \in \mathbb{Z} \) then \( \psi : \Gamma \rightarrow \mathbb{C} \) is a character

If \( k \in \frac{1}{2} + \mathbb{Z} \) and \( \Gamma = \Gamma_0(q) \) then \( \psi \) is essentially the Jacobi symbol \( (d/c) \) times a character modulo \( q \).
Automorphic Forms:

\[ \Delta_k = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \frac{\partial}{\partial x} \]  

Laplace operator

\[ (\Delta_k + \lambda)f = 0, \quad \lambda = s(1-s) = \frac{1}{4} + t^2 \]  
eigenvalue

\[ s = \frac{1}{2} + it \in \mathbb{C} \]

The Eisenstein Series:

Suppose \( \Gamma \) has parabolic elements. The fixed points of parabolic elements are cusps of \( \Gamma \backslash \mathbb{H} \). Let \( \alpha \in \mathbb{R} \cup \{ \infty \} \) be a cusp.

\[ \Gamma_{\alpha} = \{ \gamma \in \Gamma ; \gamma \alpha = \alpha \} \]  
the stability group

If \( \delta \) is trivial on \( \Gamma_{\alpha} \), then \( \alpha \) is said to be singular. For every singular cusp one has

\[ E_{\alpha}(z,s) \]  
the Eisenstein series

For example, if \( \alpha = \infty \) and \( \Gamma_{\infty} = \{ \pm \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) ; \ b \in \mathbb{Z} \} \), then

\[ E_{\infty}(z,s) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \bar{\nu}(\gamma) \left( \frac{cz+d}{1cz+d} \right)^{-k} (\text{Im} \ yz)^s, \text{ for } \text{Re} s > 1. \]
Meromorphic continuation in $s \in \mathbb{C}$

Functional equation

$$E(z, s) = \mathcal{E} \cdots, E_{\omega}(z, s), \cdots$$

$$E(z, s) = \Phi(s) \cdot E(z, 1-s)$$

$$\Phi(s) = (\varphi_{\alpha \beta}(s)) \text{ the scattering matrix}$$

The entries $\varphi_{\alpha \beta}(s)$, where $\alpha, \beta$ are singular cusps, are the coefficients in the Fourier expansion

$$E_{\alpha}(z, s) = S_{\alpha \beta} y^s + \varphi_{\alpha \beta}(s) y^{1-s}$$

$$+ \sum_{n \neq 0} \varphi_{\alpha \beta}(n, s) W(4\pi |n| y) e(mx)$$

where $W(y)$ is the Whittaker function.

In the critical strip $0 \leq \sigma = \text{Re} s \leq 1$ the poles of $E_{\omega}(z, s)$ are the poles of $\varphi_{\alpha \beta}(s)$, they are simple in the segment $\frac{1}{2} < s \leq 1$, or in the strip $0 \leq s \leq \frac{1}{2}$.

In particular there are no poles on the critical line

$$\text{Re} s = \frac{1}{2}.$$
Maass Cusp Forms:

\[ f(\sigma_\alpha z) \to 0 \quad \text{as} \; y \to \infty \]

for any singular cusp \( \alpha \)

\[ (\Delta_k + \lambda)f = 0 \]

\[ f(z) = \sum_{n \neq 0} \mathfrak{p}_f(m) \, W(4\pi |m|y) \, e(mx), \quad \text{if} \; \alpha = \infty. \]

\[ W(y) = \frac{W_{km}}{2\pi n} \, \zeta(it)(y) \quad \text{if} \; \lambda = s(1-s) \]

\[ s = \frac{1}{2} + it \]

\[ \mathfrak{p}_f(m) - \text{the Fourier coefficients} \]
**Spectral Theorem.** Let $L^2_k(\Gamma)$ be the space of automorphic functions of weight $k$, square integrable with respect to the inner product
\[
<f, g> = \int_{\Gamma \setminus \mathbb{H}} f(z) \overline{g(z)} \, d\mu(z), \quad d\mu(z) = y^2 \, dx \, dy.
\]

Let $C_k(\Gamma)$ be the linear subspace spanned by cusp forms and $R_k(\Gamma)$ be the linear subspace spanned by the residues of the Eisenstein series $E_k(z, s)$ at the poles in $\frac{1}{2} < s \leq 1$. These are orthogonal subspaces. The Laplace operator $\Delta_k$ is self-adjoint, and it has purely point spectrum in $C_k(\Gamma) \cup R_k(\Gamma)$,
\[
\frac{|k|}{2} \left( 1 - \frac{|k|}{2} \right) \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots
\]

In the orthogonal complement, say $E_k(\Gamma)$, the Laplace operator has purely continuous spectrum which covers the segment
\[
\frac{1}{4} \leq \lambda < \infty
\]
uniformly with multiplicity equal to the number of singular cusps. The eigenpacket of continuous spectrum consists of the Eisenstein series $E(z, s)$ on the critical line $s = \frac{1}{2} + it$, the spectral measure being $\frac{dt}{4\pi}$. 
The classical (holomorphic) cusp forms lie at the bottom of the spectrum. Precisely an $f \in \mathcal{C}_k(\Gamma)$ with $k \geq 0$ has the Laplace eigenvalue $\lambda = \frac{k}{2} (1 - \frac{k}{2})$ if and only if

$$F(z) = y^{-\frac{k}{2}} f(z) \in S_k(\Gamma),$$

that is

$$F(yz) = y(\gamma) (cz+d)^k F(z), \quad \gamma \in \Gamma$$

$F(z)$ holomorphic on $\mathbb{H}$

$F(z)$ vanishes at cusps

The eigenspaces $\mathcal{C}_k(\Gamma, \lambda)$ with $\lambda = s(1-s)$ are finite dimensional.

If $s$ is not real then there is an isometry

$$K_k : \mathcal{C}_k(\Gamma, \lambda) \to \mathcal{C}_{k+2}(\Gamma, \lambda)$$

given by a certain linear differential operator $K_k$ (due to H. Maass).

$$K_k = \frac{k}{2} + y (i \frac{\partial}{\partial x} + \frac{\partial}{\partial y})$$

**Questions:**
- Do Maass cusp forms exist?
- How large are the eigenspaces $\mathcal{C}_k(\Gamma, \lambda)$?
- How many there are small eigenvalues?
3. SELBERG TRACE FORMULA

Let $k = 0$ and $\nu = 1$
$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ point spectrum

$\lambda_j = s_j (1-s_j), \quad s_j = \frac{1}{2} + it_j$

\[ s_j \quad \frac{1}{2} \quad s_0 = 1 \]

$h(t)$ - holomorphic in $|\text{Im} \ t| \leq \frac{1}{2} + \varepsilon$
$h(t) = h(-t), \quad h(t) \ll (|t| + 1)^{-2 - \varepsilon}$

TRACE FORMULA

\[ \sum_j h(t_j) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \frac{e^t}{\varphi' \left( \frac{1}{2} + it \right)} \, dt \]

\[ = \frac{\text{vol}(\Gamma \backslash \text{H})}{4\pi} \int_{-\infty}^{\infty} h(t) t \tanh(\pi t) \, dt \]

\[ + \sum_{P} \frac{g(\log P)}{P^{1/2} - P^{-1/2}} \log P + \ldots \]

Here $\varphi(s) = \det \Phi(s)$, $g(x)$ is the Fourier transform of $h(t)$,
$P$ runs over the conjugacy classes of hyperbolic elements
of $\Gamma$ and $P = N_{P} \bar{P}$ denotes the norm of $P$, so $\log P$
is the length of the corresponding closed geodesic.
Weyl's law:

\[ N_\Gamma(T) = \# \{ j ; |\tau_j| \leq T \} \ll T^2 \]

\[ M_\Gamma(T) = \frac{1}{4\pi} \int_{-T}^{T} \frac{\psi'(\frac{1}{2} + it)}{\psi(\frac{1}{2} + it)} \, dt \ll T^2 \]

\[ N_\Gamma(T) + M_\Gamma(T) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} T^2 - \frac{k}{2\pi} T \log T + c_T T + O\left(\frac{T}{\log T}\right) \]

\[ M_\Gamma(T) = \# \{ \text{poles of } \psi(s) \text{ in } |\tau| \leq T \} + O(T). \]

For congruence groups we have

\[ M_\Gamma(T) \ll T \log T, \]

\[ N_\Gamma(T) \sim \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} T^2. \]

Hence there are infinitely many linearly independent Maass cusp forms.
Deformation Theory (Phillips-Sarnak):

Let

$$u_j(z) = \frac{1}{\Gamma_j} \sum_{n \neq 0} \lambda_j(n) K_{it_j}((2\pi/n)y)e(nx)$$

be a Maass cusp form of weight zero for $\Gamma = \Gamma_0(q)$ and the Laplace eigenvalue $\lambda_j = \sigma_j (1-\sigma_j)$, $\sigma_j = \frac{x}{2} + it_j$, which is a Hecke form

$$T_n u_j = \lambda_j(n) u_j \quad \text{for all } n \geq 1.$$ 

Let

$$L(s, f \otimes u_j) = \sum_{n=1}^{\infty} a(n) \lambda_j(n) n^{-s}$$

be the Rankin-Selberg $L$-function, where $a(n)$ are the Fourier coefficients of a suitable classical cusp form $f(z)$ of weight four. If $L(s_j, f \otimes u_j) \neq 0$ then the cusp form $u_j(z)$ can be destroyed by a suitable deformation of the group $\Gamma = \Gamma_0(q)$.

The non-vanishing condition was established by Deshongers - Fanouric for infinitely many cusp forms. Recently W. Luo got this for a positive density of cusp forms

$$\# \{ j; |t_j| < T, L(s_j, f \otimes u_j) \neq 0 \} \gg T^2.$$ 

This result shows that Weyl's law cannot hold for cuspidal spectrum alone. Moreover it provides an interesting zeta-function $\varphi(s)$ which is meromorphic of order two.
Closed Geodesic Theorem:
\[ \sum_{P \leq X} \log P = X - \sum_{\frac{1}{2} < \varepsilon_j < 1} \frac{X^{\varepsilon_j}}{\varepsilon_j} + O(X^{\frac{3}{4}}). \]

If \( \Gamma = \Gamma_0(q) \) then
\[ \sum_{P \leq X} \log P = X + O(X^{\frac{7}{10} + \varepsilon}) \quad (Luo + Sarnak) \]

This improvement requires cancellation of terms in the sum
\[ \sum \frac{X^{\varepsilon_j}}{\varepsilon_j}, \quad \varepsilon_j = \frac{1}{2} + it_j. \]

A strong estimate for the Rankin–Selberg L-functions is needed.

Small Eigenvalues:

\( \Gamma \subset SL_2(\mathbb{R}), \quad \Gamma \backslash \Gamma \) compact, smooth, genus \( g \geq 1 \).
\( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_j \sim 4\pi j / \text{vol}(\Gamma \backslash \Gamma) \) by Weyl's law

\( \lambda_{2g-3} \) can be arbitrarily small.

\( \lambda_{4g-2} \geq \frac{1}{4} \) always.
**Conjecture (A. Selberg).** If $\Gamma$ is a congruence group,

\[ \lambda_1 \geq \frac{1}{4}. \]

We have:

\[ \lambda_1 \geq \frac{3}{16} \quad \text{(due to Selberg, Weil's bound for Kloosterman sums)} \]

\[ \lambda_1 \geq \frac{1}{4} - \left( \frac{7}{64} \right)^2 \quad \text{(due to H. Kim & P. Sarnak, modularity of the symmetric cube representations)} \]

**Density Theorem.** If $\Gamma = \Gamma_0(q)$ then for any $\delta > \frac{1}{2}$

\[ \# \{ j > 0 ; \ s_j > \delta \} \ll q^{3 - 4\delta + \epsilon}. \]

**Problems of Multiplicity of Spectra:**

\[ \dim \mathcal{C}_k(\Gamma, \lambda) \ll \text{vol}(\Gamma \backslash \mathcal{H})(1 + 1)^{\frac{1}{2}} \]

(by trace formula)

**Question.** For $k = 0$ and $\Gamma = \Gamma_0(q)$, are the eigenvalues of $\Delta$ all simple?

For $k \geq 2$ integer, $\Gamma = \Gamma_0(q)$, $\chi(\mod{q})$, $\chi(-1) = (-1)^k$ we have

\[ \dim S_k(\Gamma, \chi) = kq \prod \frac{1 + \frac{1}{p}}{1 - \frac{1}{p}}. \]
4. CUSP FORMS OF WEIGHT ONE

$h = 1$, $\Gamma = \Gamma_0(q)$, $\chi(\text{mod } q)$ primitive character, $\chi(1) = -1$.

$\lambda_0 = \frac{1}{4} \leq \lambda_1 \leq \ldots$ point spectrum

The eigenvalue $\frac{1}{4}$ is not isolated. The space $S_1(\Gamma, \chi)$ lies at the bottom of the continuous spectrum. One cannot pick up $S_1(\Gamma, \chi)$ by a holomorphic test function in the trace formula (uncertainty principle of harmonic analysis).

![Circle with points labeled $S_0 = \frac{1}{2}$ and $S_1$]

$\dim S_1(\Gamma, \chi) \ll q^{1 + \varepsilon}$ (the trivial bound)

**Theorem** (Deligne–Serre, 1974). Every Hecke form $f \in S_1(\Gamma, \chi)$ comes from an irreducible two-dimensional Galois representation $\rho : \text{Gal}(L/\mathbb{Q}) \to GL_2(\mathbb{C})$.

By this theorem, the Hecke cusp forms of weight one can be classified as being:
- dihedral (Dih)
- tetrahedral (A₄)
- octahedral (S₄)
- icosahedral (A₅)
Suppose
\[ q = p \equiv 3 \pmod{4} \text{ is prime} \]
\[ \chi(m) = \left(\frac{m}{p}\right) \text{ the Legendre symbol} \]

\[ K = \mathbb{Q}(\sqrt{-p}) \]
\[ \mathcal{O}(K) \text{ the ring of integers} \]
\[ \text{Cl}(K) \text{ the ideal class group} \]
\[ h = h(K) = |\text{Cl}(K)| \text{ the class number} \]

For any \( \psi \in \text{Cl}(K) \), \( \psi \neq 1 \) we have a Hecke cusp form
\[ f_{\psi}(z) = \sum_{0 \neq \mathfrak{a} \in \mathcal{O}(K)} \psi(\mathfrak{a}) e(\pi \mathfrak{a} z) \in S_1(\Gamma, \chi) \]

All \( \psi \neq 1 \) are complex (no genus character)
\[ f_{\psi}(z) = f_{\bar{\psi}}(z) \text{ (no other linear relations)} \]

Hence
\[ \dim S_1(\Gamma, \chi) \geq \frac{h-1}{2} \gg p^{\frac{1}{2} - \varepsilon} \]

**Conjecture** (J.-P. Serre).
\[ \dim S_1(\Gamma, \chi) = \frac{h-1}{2} + O(p^\varepsilon) \]

**Theorem** (W. Duke)
\[ \dim S_1(\Gamma, \chi) \ll p^{\frac{11}{12} + \varepsilon} \]
The proof by Duke takes advantage of two conflicting properties of the Hecke-Fourier coefficients of non-dihedral cusp forms

\[ f(z) = \sum_{1}^{\infty} \lambda_f(m) e(mz) \in S_{1}(\Gamma, \chi) \]

The first property is the approximate orthogonality (a large sieve type inequality)

\[ \sum_{f \in B} \sum_{m \leq N} |a_n| \lambda_f(m)^2 \ll (p+N) \sum_{n \leq N} |a_n|^2 \]

where \( B \) is the Hecke basis and \( a_n \) are any complex numbers.

The second property is the boundedness of number of values \( \lambda_f(x^2) \), for example

\[ \lambda_f(x^2) \left( \frac{x}{p} \right) = 0, \pm 1, 3 \]

if \( f \) comes from octahedral representation. Hence

\[ \dim S_{oct} < \frac{7}{8} + \varepsilon \]

Similarly

\[ \dim S_{ico} < \frac{71}{12} + \varepsilon \]
5. Petersson - Kuznetsov Formulas

Petersson's Formula. Let \( \Gamma = \Gamma_0(q) \), \( \chi(\text{mod } q) \), \( \chi(-1) = (-1)^k \), \( k \geq 2 \). Let \( B_k \) be an orthogonal basis of \( S_k(\Gamma, \chi) \). For any \( f \in B_k \) let

\[
f(z) = \sum_{1}^{\infty} \lambda_f(n) n^{k/2} e(mz) \]

\[
\omega_f = (4\pi)^{1-k} \frac{\Gamma(k-1)}{\langle f, f \rangle} \approx \left( kqL(1, \text{sym}^2 f) \right)^{-1}
\]

Then for any \( m, n \geq 1 \) we have

\[
\sum_{f \in B_k} \omega_f \lambda_f(m) \lambda_f(n) = S(m, n) + 2\pi^k \sum_{c=0(q)} c^{-1} S_y(m, n; c) F(\frac{4\pi \sqrt{mn}}{c})
\]

where \( J_{k-1}(x) \) is the Bessel function and \( S_y(m, n; c) \) is the Kloosterman sum

\[
S_y(m, n; c) = \sum_{ad \equiv 1(\text{mod } c)} \chi(a)e\left(\frac{am + dn}{c}\right)
\]

A similar formula holds for Maass cusp forms (due to N.V. Kuznetsov). Let

\[
S = \sum_{c \equiv 0(q)} c^{-1} S(m, n; c) F(\frac{4\pi \sqrt{mn}}{c})
\]

where \( F(x) \) is a smooth function compactly supported on \( \mathbb{R}^+ \).
Kuznetsov's Formula. Let $\Gamma = \Gamma_0 (q)$. Let \( \{ f_j (z) \} \) be an orthonormal basis of Maass cusp forms of weight zero, and \( \mathcal{B}_k \) an orthonormal basis of holomorphic cusp forms \( f(z) \) of weight \( k \) (\( k \)-even)

\[
f_j (z) = \sqrt{q} \sum_{m \neq 0} \rho_j (m) K_{it_j} (2\pi m n y) e(mx),
\]

\[
f(z) = \sum_{m=1}^{\infty} \lambda_f (m) n^{\frac{k-1}{2}} e(mz).
\]

Then for any \( m, n \geq 1 \) we have

\[
S = \sum_{j=1}^{\infty} M(t_j) \overline{\rho_j (m)} \rho_j (m) + \text{(continuous spectrum integrals)}
\]

\[+ \sum_{k \text{ even}} N(k) \sum_{f \in \mathcal{B}_k} \overline{\lambda_f (m)} \lambda_f (m) \]

where \( M(t) \) and \( N(k) \) are given by the following integrals

\[
M(t) = \frac{\pi i}{\sinh (2\pi t)} \int_0^\infty \left( J_{\frac{t}{2}} (x) - \frac{J_{\frac{-t}{2}} (x)}{x} \right) F(x) \frac{dx}{x}
\]

\[
N(k) = \frac{4 \Gamma(k)}{(4\pi i)^k} \int_0^\infty J_{k-1} (x) F(x) \frac{dx}{x}
\]
Using estimates for the lowest eigenvalue, W. Luo, Z. Rudnick, and P. Sarnak derived

$$
\sum_{c \leq X, c \equiv a(q)} S(m, n; c) \ll X^{\frac{2}{5} + \varepsilon}
$$

Compare this with the trivial estimate $O(X^{\frac{1}{2} + \varepsilon})$ which follows by Weil's bound for individual Kloosterman sums

$$
S(m, n; c) \ll c^{\frac{1}{2} + \varepsilon}.
$$

**PROBLEM.** Show that $S(m, n; p)$ changes sign infinitely often as $p$ runs over primes.

**CONJECTURE** (N. Katz). The angles $\Theta_p$ of Kloosterman sums

$$
S(m, n; p) = 2 \sqrt{p} \cos \Theta_p, \quad 0 \leq \Theta_p < \pi
$$

are equidistributed (as $p \to \infty$) with respect to the Sato–Tate measure

$$
d\mu \Theta = \frac{1}{\pi} (\sin \Theta)^2 d\Theta.
$$
6. Normalization of Cusp Forms

\[ f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz) \quad \text{Hecke cusp form} \]

\[ \lambda_f(1) = 1 \quad \text{(arithmetic)} \]

\[ \langle f, f \rangle = \int_{\Gamma \backslash \mathbb{H}} |f(z)|^2 y^k \, d\mu z = 1 \quad \text{(spectral)} \]

Suppose we have \( \lambda_f(1) = 1 \). Computing the residue of the Rankin–Selberg \( L \)-function

\[ L(s, f \otimes f) = \sum_{n=1}^{\infty} |\lambda_f(n)|^2 n^{-s} \]

at \( s = 1 \) we find that

\[ \langle f, f \rangle = \frac{2 \Gamma(k)}{\pi (4\pi)^k} \prod_{p} \prod_{l \equiv 0 \mod p^2} \left( 1 - \frac{1}{p^2} \right) L(1, \text{sym}^2 f) \]

where \( L(s, \text{sym}^2 f) \) is the \( L \)-function attached to the symmetric square representation of \( f \). We have

\[ (\log kq)^{-1} \ll L(1, \text{sym}^2 f) \ll \log kq \]

(due to J. Hoffstein and P. Lockhart). There is no exceptional zero for the symmetric square \( L \)-functions.
7. EQUIDISTRIBUTION OF ROOTS OF QUADRATIC CONGRUENCES

Let \( f(x) = aX^2 + bX + c \in \mathbb{Z}[X] \), irreducible. The roots of
\[
\sum f(\gamma) \equiv 0 \pmod{p}
\]
are equidistributed modulo \( p \), precisely.

**THEOREM (W. Duke + H.I. + A. Toth).** For any continuous function \( F(t) \) periodic of period one we have
\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} F\left(\frac{\gamma}{p}\right) = \int_0^1 F(t) \, dt.
\]

The proof exploits the spectral theory of automorphic forms in its full force (together with the density theorem for small eigenvalues). Moreover a sieve method is used to produce sums over primes.