SPECTRAL THEORY OF AUTOMORPHIC FORMS AND ANALYTIC NUMBER THEORY

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1. INTRODUCTION

I shall speak mostly about the GL_2 forms, i.e. functions on the upper half-plane

$$\mathbb{H} = SL_2(\mathbb{R})/K = \{ z = x+iy \,;\, x \in \mathbb{R},\, y \in \mathbb{R}^+ \}.$$

Tools
- Combinatorial identities of sieve type
- Summation formulas of trace type

Theta series

$$\Theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z), \quad e(z) = e^{2\pi i z}$$

$$\Theta(\gamma z) = \vartheta(\gamma)(cz+d)^{\frac{1}{2}}\,\Theta(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4).$$

Poisson's formula

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

2. SPECTRAL THEORY

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \; ; \; c \equiv 0 \pmod{q} \right\}$$

$$[\Gamma_0(1) : \Gamma_0(q)] = q \prod_{p \mid q} \left(1 + \frac{1}{p}\right)$$

Let $\Gamma \subset SL_2(\mathbb{R})$ a group acting discontinuously on $\mathbb{H}$

$$\gamma z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Adelic setting, Hecke operators for congruence groups.

Automorphic Functions :

$$f : \mathbb{H} \to \mathbb{C}$$

$k \in \mathbb{R}$ weight

$\vartheta : \Gamma \to \mathbb{C}$ multiplier

$$f(\gamma z) = \vartheta(\gamma) \left(\frac{cz+d}{|cz+d|} \right)^k f(z)$$

$|\vartheta(\gamma)| = 1$

$f(z)$ - polynomial growth at cusps

If $k \in \mathbb{Z}$ then $\vartheta : \Gamma \to \mathbb{C}$ is a character

If $k \in \frac{1}{2} + \mathbb{Z}$ and $\Gamma = \Gamma_0(q)$ then ϑ is essentially the Jacobi symbol $\left(\frac{d}{c}\right)$ times a character modulo q.

Automorphic Forms :

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}$$ Laplace operator

$$(\Delta_k + \lambda)f = 0, \quad \lambda = s(1-s) = \tfrac{1}{4} + t^2$$ eigenvalue

$$s = \tfrac{1}{2} + it \in \mathbb{C}$$

The Eisenstein Series :

Suppose Γ has parabolic elements.
The fixed points of parabolic elements are cusps of $\Gamma\backslash\mathbb{H}$.
Let $\mathfrak{a} \in \mathbb{R} \cup \{\infty\}$ be a cusp

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma ; \gamma\mathfrak{a} = \mathfrak{a}\}$$ the stability group

If ϑ is trivial on $\Gamma_{\mathfrak{a}}$ then $\mathfrak{a}$ is said to be singular.
For every singular cusp one has

$$E_{\mathfrak{a}}(z,s)$$ the Eisenstein series

For example, if $\mathfrak{a} = \infty$ and $\Gamma_\infty = \{\pm\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) ; b \in \mathbb{Z}\}$, then

$$E_\infty(z,s) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \bar{\vartheta}(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^{-k}(\operatorname{Im}\gamma z)^s, \quad \text{for } \operatorname{Re} s > 1.$$

Meromorphic continuation in $s \in \mathbb{C}$
Functional equation

$$\mathcal{E}(z,s) = [\ldots, E_{\mathfrak{a}}(z,s), \ldots]^t$$

$$\mathcal{E}(z,s) = \Phi(s)\,\mathcal{E}(z,1-s)$$

$$\Phi(s) = (\varphi_{\mathfrak{a}\mathfrak{b}}(s))$$ the scattering matrix

The entries $\varphi_{\mathfrak{a}\mathfrak{b}}(s)$, where $\mathfrak{a}, \mathfrak{b}$ are singular cusps, are the coefficients in the Fourier expansion

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a}\mathfrak{b}}\, y^{s} + \varphi_{\mathfrak{a}\mathfrak{b}}(s)\, y^{1-s}$$
$$+ \sum_{n \neq 0} \varphi_{\mathfrak{a}\mathfrak{b}}(n,s)\, W(4\pi|n|y)\, e(nx)$$

where $W(y)$ is the Whittaker function.

In the critical strip $0 \leq \sigma = \operatorname{Re} s \leq 1$ the poles of $E_{\mathfrak{a}}(z,s)$ are the poles of $\varphi_{\mathfrak{a}\mathfrak{b}}(s)$, they are simple in the segment $\frac{1}{2} < s \leq 1$, or in the strip $0 \leq \sigma < \frac{1}{2}$.

In particular there are no poles on the critical line

$$\operatorname{Re} s = \frac{1}{2}.$$

Maass Cusp Forms :

$$f(\sigma_{\mathfrak{a}} z) \to 0 \quad \text{as } y \to \infty$$
for any singular cusp $\mathfrak{a}$

$$(\Delta_k + \lambda) f = 0$$

$$f(z) = \sum_{n \neq 0} \rho_f(n) W(4\pi |n| y) e(nx), \quad \text{if } \mathfrak{a} = \infty.$$

$$W(y) = W_{\frac{kn}{2|n|}, it}(y) \quad \text{if } \lambda = s(1-s)$$
$$s = \tfrac{1}{2} + it$$

$\rho_f(n)$ – the Fourier coefficients

SPECTRAL THEOREM. Let $\mathcal{L}_k^2(\Gamma)$ be the space of automorphic functions of weight k, square integrable with respect to the inner product

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z)\bar{g}(z)\, d\mu z, \qquad d\mu z = y^{-2} dx dy.$$

Let $\mathcal{C}_k(\Gamma)$ be the linear subspace spanned by cusp forms and $\mathcal{R}_k(\Gamma)$ be the linear subspace spanned by the residues of the Eisenstein series $E_\alpha(z,s)$ at the poles in $\frac{1}{2} < s \leq 1$. These are orthogonal subspaces. The Laplace operator Δ_k is self-adjoint, and it has purely point spectrum in $\mathcal{C}_k(\Gamma) \cup \mathcal{R}_k(\Gamma)$,

$$\frac{|k|}{2}\left(1 - \frac{|k|}{2}\right) \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

In the orthogonal complement, say $\mathcal{E}_k(\Gamma)$, the Laplace operator has purely continuous spectrum which covers the segment

$$\frac{1}{4} \leq \lambda < \infty$$

uniformly with multiplicity equal to the number of singular cusps. The eigenpacket of continuous spectrum consists of the Eisenstein series $\mathcal{E}(z,s)$ on the critical line $s = \frac{1}{2} + it$, the spectral measure being $\frac{dt}{4\pi}$.

The classical (holomorphic) cusp forms lie at the bottom of the spectrum. Precisely an $f \in \mathcal{C}_k(\Gamma)$ with $k \geq 0$ has the Laplace eigenvalue $\lambda = \frac{k}{2}(1-\frac{k}{2})$ if and only if

$$F(z) = y^{-\frac{k}{2}} f(z) \in S_k(\Gamma),$$

that is

$$F(\gamma z) = \vartheta(\gamma)(cz+d)^k F(z), \quad \gamma \in \Gamma$$
$$F(z) \text{ holomorphic on } \mathbb{H}$$
$$F(z) \text{ vanishes at cusps}$$

The eigenspaces $\mathcal{C}_k(\Gamma, \lambda)$ with $\lambda = s(1-s)$ are finite dimensional.
If s is not real then there is an isometry

$$K_k : \mathcal{C}_k(\Gamma, \lambda) \to \mathcal{C}_{k+2}(\Gamma, \lambda)$$

given by a certain linear differential operator K_k (due to H. Maass).

$$K_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

QUESTIONS :

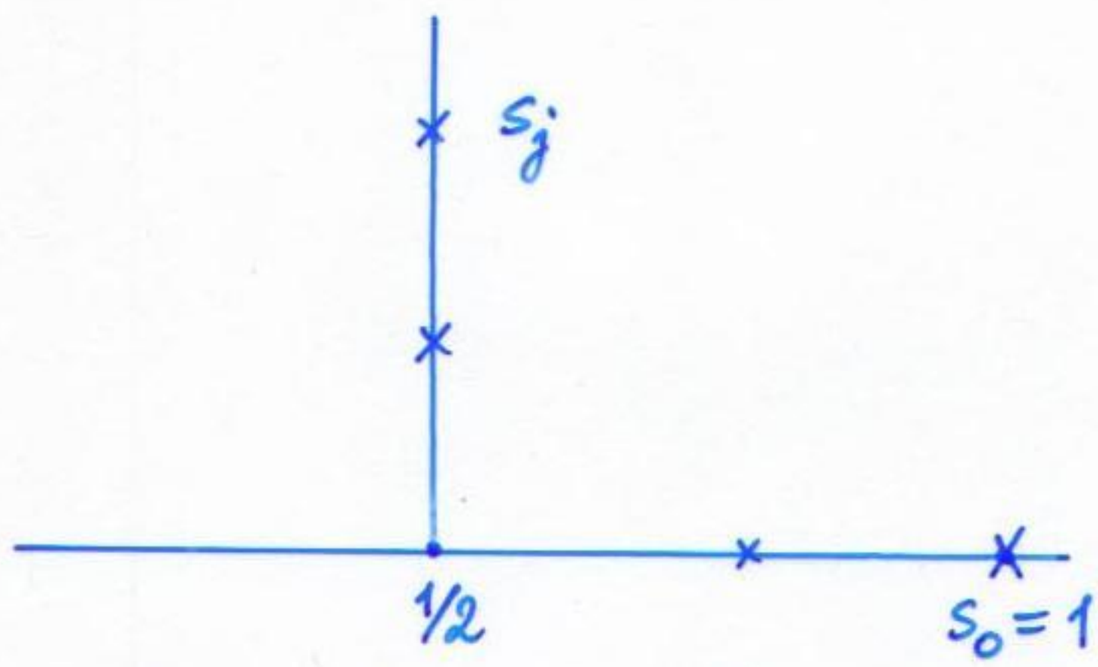
- Do Maass cusp forms exist ?
- How large are the eigenspaces $\mathcal{C}_k(\Gamma, \lambda)$?
- How many there are small eigenvalues ?

3. SELBERG TRACE FORMULA

Let $k=0$ and $\vartheta=1$

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq\dots \qquad \text{point spectrum}$$

$$\lambda_j=s_j(1-s_j), \qquad s_j=\tfrac{1}{2}+it_j$$



$h(t)$ - holomorphic in $|\operatorname{Im} t|\leq\frac{1}{2}+\varepsilon$

$h(t)=h(-t), \qquad h(t)\ll(|t|+1)^{-2-\varepsilon}$

TRACE FORMULA

$$\sum_j h(t_j)+\frac{1}{4\pi}\int_{-\infty}^{\infty}h(t)\frac{-\varphi'}{\varphi}(\tfrac{1}{2}+it)\,dt$$

$$=\frac{\operatorname{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{-\infty}^{\infty}h(t)\,t\tanh(\pi t)\,dt$$

$$+\sum_{\mathcal{P}}\frac{g(\log P)}{P^{1/2}-P^{-1/2}}\log P \quad+\dots$$

Here $\varphi(s)=\det\Phi(s)$, $g(x)$ is the Fourier transform of $h(t)$, $\mathcal{P}$ runs over the conjugacy classes of hyperbolic elements of Γ and $P=N\mathcal{P}$ denotes the norm of $\mathcal{P}$, so $\log P$ is the length of the corresponding closed geodesic.

Weyl's law:

$$N_\Gamma(T) = \#\{j\,;\, |t_j| \le T\} \ll T^2$$

$$M_\Gamma(T) = \frac{1}{4\pi}\int_{-T}^{T} \frac{-\varphi'}{\varphi}\left(\tfrac{1}{2}+it\right) dt \ll T^2$$

$$N_\Gamma(T) + M_\Gamma(T) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2 - \frac{h}{\pi} T \log T + c_\Gamma T + O\left(\frac{T}{\log T}\right)$$

$$M_\Gamma(T) = \#\{\text{poles of } \varphi(s) \text{ in } |t| \le T\} + O(T).$$

For congruence groups we have

$$M_\Gamma(T) \ll T \log T,$$

$$N_\Gamma(T) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2.$$

Hence there are infinitely many linearly independent Maass cusp forms.

Deformation Theory (Phillips-Sarnak):

Let
$$u_j(z) = \sqrt{y} \sum_{n \neq 0} \lambda_j(n) K_{it_j}(2\pi|n|y)\, e(nx)$$

be a Maass cusp form of weight zero for $\Gamma = \Gamma_0(q)$ and the Laplace eigenvalue $\lambda_j = s_j(1-s_j)$, $s_j = \frac{1}{2} + it_j$, which is a Hecke form

$$T_n u_j = \lambda_j(n) u_j \quad \text{for all } n \geq 1.$$

Let
$$L(s, f \otimes u_j) = \sum_1^\infty a(n) \lambda_j(n) n^{-s}$$

be the Rankin-Selberg L-function, where $a(n)$ are the Fourier coefficients of a suitable classical cusp form $f(z)$ of weight four. If $L(s_j, f \otimes u_j) \neq 0$ then the cusp form $u_j(z)$ can be destroyed by a suitable deformation of the group $\Gamma = \Gamma_0(q)$.

The non-vanishing condition was established by Deshouillers - Iwaniec for infinitely many cusp forms. Recently W. Luo got this for a positive density of cusp forms

$$\#\{j;\ |t_j| < T,\ L(s_j, f \otimes u_j) \neq 0\} \gg T^2.$$

This result shows that Weyl's law cannot hold for cuspidal spectrum alone. Moreover it provides an interesting zeta-function $\varphi(s)$ which is meromorphic of order two.

Closed Geodesic Theorem :

$$\sum_{P \le X} \log P = X - \sum_{1/2 < s_j < 1} \frac{X^{s_j}}{s_j} + O(X^{\frac{3}{4}}).$$

If $\Gamma = \Gamma_0(q)$ then

$$\sum_{P \le X} \log P = X + O(X^{\frac{7}{10}+\varepsilon}) \quad \text{(Luo + Sarnak)}$$

This improvement requires cancellation of terms in the sum

$$\sum_{|t_j| \le T} \frac{X^{s_j}}{s_j}, \qquad s_j = \tfrac{1}{2} + i t_j$$

A strong estimate for the Rankin-Selberg L-functions is needed.

Small Eigenvalues :

$\Gamma \subset SL_2(\mathbb{R})$, $\Gamma \backslash \mathbb{H}$ compact, smooth, genus $g > 1$.
$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$, $\lambda_j \sim 4\pi j / \mathrm{vol}(\Gamma \backslash \mathbb{H})$
by Weyl's law

λ_{2g-3} can be arbitrarily small

$\lambda_{4g-2} \geqslant \frac{1}{4}$ always

CONJECTURE (A. Selberg). If Γ is a congruence group

$$\lambda_1 \geq \frac{1}{4}.$$

We have:

$\lambda_1 \geq \frac{3}{16}$ (due to Selberg, Weil's bound for Kloosterman sums)

$\lambda_1 \geq \frac{1}{4} - \left(\frac{7}{64}\right)^2$ (due to H. Kim & P. Sarnak, modularity of the symmetric cube representations)

DENSITY THEOREM. If $\Gamma = \Gamma_0(q)$ then for any $\sigma > \frac{1}{2}$

$$\#\{j > 0;\ s_j > \sigma\} \ll q^{3-4\sigma+\varepsilon}.$$

PROBLEMS OF MULTIPLICITY OF SPECTRA:

$$\dim \mathcal{C}_k(\Gamma, \lambda) \ll \mathrm{vol}(\Gamma\backslash\mathbb{H})(|\lambda|+1)^{\frac{1}{2}}$$

(by trace formula)

Question. For $k=0$ and $\Gamma = SL_2(\mathbb{Z})$ are the eigenvalues of Δ all simple?

For $k \geq 2$ integer, $\Gamma = \Gamma_0(q)$, $\chi \pmod q$, $\chi(-1) = (-1)^k$ we have

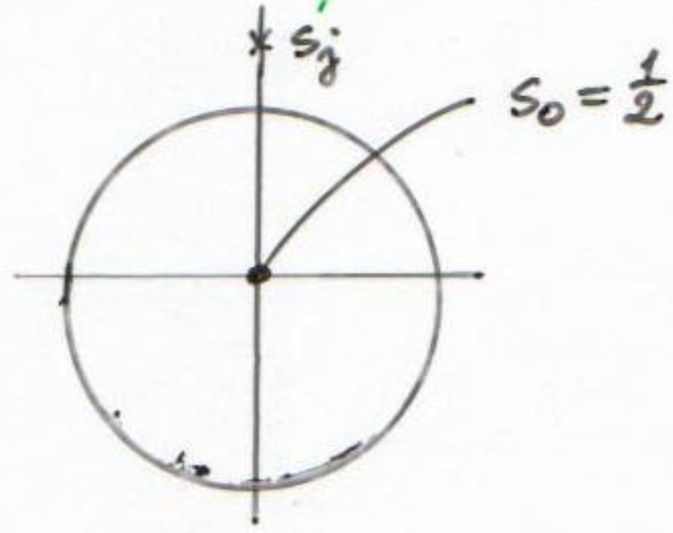
$$\dim S_k(\Gamma, \chi) \asymp kq \prod_{p|q} \left(1 + \frac{1}{p}\right).$$

4. CUSP FORMS OF WEIGHT ONE

$k=1$, $\Gamma=\Gamma_0(q)$, $\chi \pmod q$ primitive character, $\chi(-1)=-1$.

$$\lambda_0=\frac{1}{4}\leq\lambda_1\leq\dots \qquad \text{point spectrum}$$

The eigenvalue $\frac{1}{4}$ is not isolated
The space $S_1(\Gamma,\chi)$ lies at the bottom of the continuous spectrum. One cannot pick up $S_1(\Gamma,\chi)$ by a holomorphic test function in the trace formula (uncertainity principle of harmonic analysis)



$$\dim S_1(\Gamma,\chi)\ll q^{1+\varepsilon}$$

(the trivial bound)

THEOREM (Deligne-Serre, 1974). Every Hecke form $f\in S_1(\Gamma,\chi)$ comes from an irreducible two-dimensional Galois representation $\rho: Gal(L/\mathbb{Q})\to GL_2(\mathbb{C})$.

By this theorem the Hecke cusp forms of weight one can be classified as being
- dihedral (D_n)
- tetrahedral (A_4)
- octahedral (S_4)
- icosahedral (A_5)

Suppose

$q = p \equiv 3 \pmod 4$ is prime

$\chi(n) = \left(\frac{n}{p}\right)$ the Legendre symbol

$K = \mathbb{Q}(\sqrt{-p})$

$\mathcal{O}(K)$ the ring of integers

$\mathcal{C}l(K)$ the ideal class group

$h = h(K) = |\mathcal{C}l(K)|$ the class number

For any $\psi \in \hat{\mathcal{C}l}(K)$, $\psi \neq 1$ we have a Hecke cusp form

$$f_\psi(z) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}(K)} \psi(\mathfrak{a}) e(zN\mathfrak{a}) \in S_1(\Gamma, \chi)$$

All $\psi \neq 1$ are complex (no genus character)

$f_\psi(z) = f_{\bar{\psi}}(z)$ (no other linear relations)

Hence

$$\dim S_1(\Gamma, \chi) \geq \frac{h-1}{2} \gg p^{\frac{1}{2}-\varepsilon}.$$

CONJECTURE (J.-P. Serre).

$$\dim S_1(\Gamma, \chi) = \frac{h-1}{2} + O(p^{\varepsilon})$$

THEOREM (W. Duke)

$$\dim S_1(\Gamma, \chi) \ll p^{\frac{11}{12}+\varepsilon}$$

The proof by Duke takes advantage of two conflicting properties of the Hecke-Fourier coefficients of non-dihedral cusp forms

$$f(z) = \sum_{1}^{\infty} \lambda_f(n) e(nz) \in S_1(\Gamma, \chi)$$

The first property is the approximate orthogonality (a large sieve type inequality)

$$\sum_{f \in \mathcal{B}} \Big| \sum_{n \le N} a_n \lambda_f(n) \Big|^2 \ll (p+N) \sum_{n \le N} |a_n|^2$$

where $\mathcal{B}$ is the Hecke basis and a_n are any complex numbers.

The second property is the boundedness of number of values $\lambda_f(\ell^2)$, for example

$$\lambda_f(\ell^2)\left(\frac{\ell}{p}\right) = 0, \pm 1, 3$$

if f comes from octahedral representation. Hence

$$\dim S_{oct} \ll p^{\frac{7}{8}+\varepsilon}$$

Similarly

$$\dim S_{ico} \ll p^{\frac{11}{12}+\varepsilon}.$$

5. PETERSSON – KUZNETSOV FORMULAS

PETERSSON'S FORMULA. Let $\Gamma = \Gamma_0(q)$, $\chi \pmod q$, $\chi(-1) = (-1)^k$, $k \geq 2$. Let $\mathcal{B}_k$ be an orthogonal basis of $S_k(\Gamma, \chi)$. For any $f \in \mathcal{B}_k$ let

$$f(z) = \sum_1^\infty \lambda_f(n)\, n^{\frac{k-1}{2}} e(nz)$$

$$\omega_f = (4\pi)^{1-k}\, \Gamma(k-1)/\langle f, f\rangle \asymp \left(kq\, L(1, \mathrm{sym}^2 f)\right)^{-1}$$

Then for any $m, n \geq 1$ we have

$$\sum_{f \in \mathcal{B}_k} \omega_f\, \bar{\lambda}_f(m) \lambda_f(n) = \delta(m,n) + 2\pi i^k \sum_{c \equiv 0(q)} c^{-1} S_\chi(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where $J_{k-1}(x)$ is the Bessel function and $S_\chi(m,n;c)$ is the Kloosterman sum

$$S_\chi(m,n;c) = \sum_{ad \equiv 1 (\mathrm{mod}\, c)} \chi(a) e\left(\frac{am + dn}{c}\right)$$

A similar formula holds for Maass cusp forms (due to N.V. Kuznetsov). Let

$$S = \sum_{c \equiv 0(q)} c^{-1} S(m,n;c)\, F\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where $F(x)$ is a smooth function compactly supported on $\mathbb{R}^+$.

KUZNETSOV'S FORMULA. Let $\Gamma = \Gamma_0(q)$. Let $\{f_j(z)\}$ be an orthonormal basis of Maass cusp forms of weight zero, and $\mathcal{B}_k$ an orthonormal basis of holomorphic cusp forms $f(z)$ of weight k (k-even)

$$f_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{it_j}(2\pi |n| y) e(nx),$$

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz).$$

Then for any $m, n \geq 1$ we have

$$S = \sum_{j=1}^{\infty} \mathcal{M}(t_j) \bar{\rho}_j(m) \rho_j(n) + (\text{continuous spectrum integrals})$$
$$+ \sum_{k \text{ even}} \mathcal{N}(k) \sum_{f \in \mathcal{B}_k} \bar{\lambda}_f(m) \lambda_f(n)$$

where $\mathcal{M}(t)$ and $\mathcal{N}(k)$ are given by the following integrals

$$\mathcal{M}(t) = \frac{\pi i}{\sinh(2\pi t)} \int_0^{\infty} \left(J_{2it}(x) - J_{-2it}(x) \right) F(x) \frac{dx}{x}$$

$$\mathcal{N}(k) = \frac{4\Gamma(k)}{(4\pi i)^k} \int_0^{\infty} J_{k-1}(x) F(x) \frac{dx}{x}$$

Using estimates for the lowest eigenvalue W. Luo, Z. Rudnick and P. Sarnak derived

$$\sum_{\substack{c \le X \\ c \equiv a(q)}} c^{-1} S(m,n;c) \ll X^{\frac{2}{5}+\varepsilon}$$

Compare this with the trivial estimate $O(X^{\frac{1}{2}+\varepsilon})$ which follows by Weil's bound for individual Kloosterman sums

$$S(m,n;c) \ll c^{\frac{1}{2}+\varepsilon}.$$

PROBLEM. Show that $S(m,n;p)$ changes sign infinitely often as p runs over primes.

CONJECTURE (N. Katz). The angles θ_p of Kloosterman sums

$$S(m,n;p) = 2\sqrt{p}\cos\theta_p, \qquad 0 \le \theta_p < \pi$$

are equidistributed (as $p \to \infty$) with respect to the Sato-Tate measure

$$d\mu\theta = \frac{2}{\pi}(\sin\theta)^2\, d\theta.$$

6. NORMALIZATION OF CUSP FORMS

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)\, n^{\frac{k-1}{2}} e(nz) \qquad \text{Hecke cusp form}$$

$$\lambda_f(1) = 1 \qquad \text{(arithmetic)}$$

$$\langle f, f \rangle = \int_{\Gamma \backslash \mathbb{H}} |f(z)|^2 y^k \, d\mu z = 1 \qquad \text{(spectral)}$$

Suppose we have $\lambda_f(1) = 1$. Computing the residue of the Rankin - Selberg L-function

$$L(s, f \otimes f) = \sum_{n=1}^{\infty} |\lambda_f(n)|^2 n^{-s}$$

at $s = 1$ we find that

$$\langle f, f \rangle = \frac{2\Gamma(k)}{\pi (4\pi)^k} q \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) L(1, \mathrm{sym}^2 f)$$

where $L(s, \mathrm{sym}^2 f)$ is the L-function attached to the symmetric square representation of f. We have

$$(\log kq)^{-1} \ll L(1, \mathrm{sym}^2 f) \ll \log kq$$

(due to J. Hoffstein and P. Lockhart). There is no exceptional zero for the symmetric square L-functions.

7. EQUIDISTRIBUTION OF ROOTS OF QUADRATIC CONGRUENCES

Let $f(X) = aX^2 + bX + c \in \mathbb{Z}[X]$, irreducible. The roots of

$$f(\nu) \equiv 0 \pmod{p}$$

are equidistributed modulo p, precisely

THEOREM (W. Duke + H.I. + A. Toth). For any continuous function $F(t)$ periodic of period one we have

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \le x} \sum_{f(\nu) \equiv 0 (p)} F\left(\frac{\nu}{p}\right) = \int_0^1 F(t)\,dt.$$

The proof exploits the spectral theory of automorphic forms in its full force (together with the density theorem for small eigenvalues). Moreover a sieve method is used to produce sums over primes.