CAN P-ADIC INTEGRALS BE COMPUTED?
I think so....
3 Threads:

- Tarski’s decision procedure
- p-adic integration
- motives
First thread

Tarski's decision procedure

1930 Alfred Tarski:

There is a decision procedure for sentences in the elementary theory of real closed fields.
Elementary theory:

Language contains

0, 1, +, x, (, )
∧, ∨, ¬ (and, or, not)
∀x, ∃x
x₁, x₂, x₃, …
=, >

Does not contain

∀n ∈ ℤ
{x₁, x₂, x₃, …}
π, e, ln2
cos x
∫ f dx …
Tarski's method is a mechanical procedure for the elimination of quantifiers (Q.E.)

Example:
\[ a \neq 0 \land (\exists x)(ax^2 + bx + c = 0) \]
\[ \Downarrow \text{Q.E.} \]
\[ b^2 - 4ac \geq 0 \]
Example:
positive semi-definite quartic

\[(\forall x)(x^4 + px^2 + qx + r \geq 0)\]

\[\Downarrow \quad \text{Q.E.}\]

\[
[256r^3 - 128p^2r^2 + 144pq^2r + 16p^4r - 27q^4
- 4p^3q^2 \geq 0 \land
8pr - 9q^2 - 2p^3 \leq 0] \lor

[27q^2 + 8p^3 \geq 0 \land 8pr - 9q^2 - 2p^3 \geq 0] \land r \geq 0
\]
1975 George Collins found a vastly improved method of quantifier elimination. Mathematica 4.0 implements the algorithm in an experimental package.
Ax-Kochen, Ershov

Paul J. Cohen  1969

"Decision procedures for real and p-adic fields"

Later results

Denef, Macintyre, Pas, ....
on quantifier elimination for p-adic fields.
Pas's language

0, 1, +, x, (, )
∧, ∨, ⊥
∀x, ∃x
(x, x₂, x₃) (3 sorts)

= ord

(angular component)

3 sorts:
valued field
value group
residue field

Details omitted
Pas's language does not contain uniformizer sets, field extensions, Galois groups, ....

Pas 1989 (building on earlier results): the $p$-adic quantifiers can be eliminated from this language for the theory of henselian fields.

Applications to $p$-adic integration.
2nd Thread:

p-adic integration
$F$ \hspace{1cm} p-adic field
characteristic zero

$g$ \hspace{1cm} reductive Lie algebra

$x \in g$ \hspace{1cm} semisimple

Compute

$$\int_{O^*(X)} f$$

with an invariant measure
Prove identities by computing integrals?

Example $g = \text{so}(5)$
res. char $F \neq 2$, $F_q$ res. field
$|\alpha(x)| = q^{-r/2}$, $r$ odd.

$P_x$ char. polynomial

$P_x = \lambda P^0_x(\lambda)$; $P^0_x(\lambda)$ roots $\pm t_1, \pm t_2$

$R_x \in F_q[\lambda]$ roots $t_i^2/q^r$

elliptic curve $y^2 = R_x(x^2)$ over $F_q$

$E_x$
\[ \int_{O^{st}(X)} f = A(q) + B(q) \mid Ex(q) \mid \]

A, B \neq 0 rational functions

What does it mean to compute the integral?

**Answer:** Find A, B, Ex

**Wrong Answer:** For a given p-adic field and parameter X, find the number \( A(q) + B(q) \mid Ex(q) \mid \).
What made this calculation possible?

1) As $X$ varies, $E_X$ varies in a regular way inside a family of elliptic curves.

2) As the local field varies, the "same" family of elliptic curves is obtained.

Conclusion: These $p$-adic integrals could be computed because $A, B, E_X$ are global objects

$E_X \quad y^2 = x^{24} - ax^2 + b$ over $\mathbb{Q}(a,b)$
An identity needed for the trace formula
\[
\sum_{\text{O}^\text{st}(x)} f = \sum_{\text{O}^\text{st}(y)} f'
\]

so(5) \quad sp(4)

\[
A + B|E_x| = A + B|E_y|
\]

**Conclusion** This identity is true everywhere locally because of a single global identity of Chow motives:

\[E_x \sim E_y\]

over \(\mathbb{Q}(a,b)\)
Thesis The computation of a p-adic integral is an effective algorithm to obtain the underlying virtual Chow motive.

Denef-Loeser principle
(Strasbourg 2000)

All "natural" p-adic integrals are motivic.
Third thread

motives
Introduction to motivic integration

\[ \int 1 \times 1^k \, dx = \sum_{l=0}^{\infty} \frac{q^{k+l}}{1 + l \cdot 1} \int \frac{du}{1 + l \cdot 1} \]

\[ \mathbb{F}_q[[t]] \quad l = 0 \quad l \cdot 1 = 1 \]

\[ = \left( 1 + \frac{1}{q^{k+1}} + \frac{1}{q^{2(k+1)}} + \ldots \right) \left( 1 - \frac{1}{q} \right) \]

\[ \int 1 \times 1^k \, dx = \left( 1 + \frac{1}{q^{k+1}} + \ldots \right) \left( 1 - \frac{1}{q} \right) \]

\[ \mathbb{F}_q[[t]] \]

if \( k = \mathbb{F}_q \) \[ q = |A'(\mathbb{F}_q)| \]

if \( \text{char } k = 0 \) \[ q = LL = [A'] \]
$k$, char 0.

$K_0(Sch_k)$

Grothendieck ring of algebraic varieties over $k$

$[S] \quad S$ alg. variety

$[S \times S'] = [S][S']$

$[S] = [S'] \quad S, S'$ iso

$[S] = [S \cup S'] + [S'] \quad S'$ closed in $S$.

$K_0(Sch_k)_{loc}$ invert $L$
Ring \( \hat{K}_0^v(Mot_{k,\bar{q}}) \otimes \mathbb{Q} \)

\( Mot_{k,\bar{q}} \) cat. of Chow motives over \( k \)

coefficients in \( \bar{Q} \)

triples \( (S, p, n) \)

\( K_0 \) Grothendieck group

\( \hat{K}_0(\text{Sch}_k)_{\text{loc}} \)

image of \( \hat{K}_0(\text{Sch}_k)_{\text{loc}} \)

complete with respect to filtration \( \frac{S_i}{L^i} \rightarrow 0 \) if

\( \dim S_i - i \rightarrow -\infty \)
Denef-Löeser 1999:
"Definable sets, motives and p-adic integrals"

Roughly:

- Motives can be attached to formulas in Pas's language.

- The trace of Frobenius on the motive equals the p-adic integral over the set defined by the formula.
My results put orbital integrals into this framework.

$F \text{ p-adic char } 0$

Parameters $n, k, r$

$n \geq 1 \quad k \leq n$

$r \in \mathbb{Q} \quad r = \frac{\ell}{h} \quad (\ell, h) = 1.$

$G = \mathfrak{so}(2n+1)$

$h = \mathfrak{so}(2k+1) \times \mathfrak{so}(2n-2k+1)$

endoscopic
strip(r) : $X \in G \mid |\alpha(X)| = q^{-r}$ \hspace{1cm} \forall \alpha$

$P_X(\lambda) = \lambda P^0_X(\lambda)$ char. poly.

$R_X \in F_q[\lambda]$ \hspace{1cm} \text{roots } t^\lambda_i / w \in F_q$

The orbital integrals are expected to degenerate outside these strips.
Hope: If $R_x = R_x'$, then their orbital integrals are equal.

$tube = strip(r) \times \mathbb{R}$

$x \rightarrow R_x$ partitions the $strip(r)$ into tubes.
Fundamental Lemma

Conjecture (Langlands-Shelstad)

\[
\left( q^{\sigma(r)} \right) \sum_X \sum_{\mathcal{O}(X) \cap \mathcal{G}(O_F)} \delta(X) \cdot \delta(0) \cdot \mathcal{O}(Y) \times \mathcal{O}(Z) \times \mathcal{H}(0) \\
\prod_X \mathcal{P}_X \cdot \mathcal{P}_Y \cdot \mathcal{P}_Z
\]

\[\sigma(r) = r \cdot \sigma_0\]

\[\text{sgn}(X,Y,Z) \in \{0,1,-1\}\]

\(X,Y,Z\) in \text{strip}(r)\]
Proposition \(-\text{res} \cdot \text{char } F \gg 0\):
\[\text{sgn}^{-1}(x), \ x \in \{0,1,-1\}\]
is given by a formula in the language of rings.

(\text{ord, ac, } \forall n \in \mathbb{Z}, \ldots \text{ not used})

Formula for \(\text{sgn}^{-1}\{1,-1\}\):
\[\lambda P_x = P_y P_z\]
resultant \((P_x, P'_x) \neq 0\)

Formula for \(\text{sgn}^{-1}(1)\):
Start with Waldspurger's formula \((2001)\)
$\text{Denef-Loeser apparatus gives a } +1 \text{-motive } \in K_0(M_{\alpha,\bar{\alpha}})\overline{\alpha}$

$-1 \text{-motive}$

The Langlands-Shelstad transfer factor "is" a motive.
Individual orbits are not given by a formula in Pas's language:

\[ p_x \in F[x] \]

\[ \uparrow \text{can't express individual } p\text{-adic numbers} \]

But tubes are OK

\[ R_x \in F_q[x] \]

\[ \hat{R}_x \in O_k[Z][\lambda] = S[\lambda] \]

\[ Z = Z_r = \mathfrak{a}\mathfrak{a}'/\mathfrak{a}\mathfrak{a} \]

for some Lie algebra \( \mathfrak{a} \)

\[ [k:Q] < \infty \]

in \( L_{\text{pas}}(O_k[Z]) \)
Conjecture

Given $n, k, r$

$$L_{r, \sigma_0} \begin{pmatrix} \mathbb{H}_{G,+}^{G, \ast} \mathbb{O}_{n, k, r}^{G, -} \end{pmatrix} = \mathbb{H}_{n, k, r}^{H, \ast}$$

in $K_0'(M_{\mathbb{Q}(Z_r), \mathbb{Q})_{\text{loc}, \mathbb{Q}}}$

(Motivic fundamental lemma)

This single identity governs the fundamental lemma over the entire strip$(r)$ at almost all places.
Remark The Denef-Loeser companion theorem relates the trace of Frobenius on these motives to the traditional fundamental lemma.

Remark There are two local-global pathways

1) standard one used in applications of the trace formula

2) new one given by the Denef-Loeser apparatus
Problem 1 Give effective algorithms to find \( \Theta_{n,k}\)
(compute the p-adic integrals)

Problem 2 Prove the "hope": \( R_x = R_{x'} \Rightarrow \) orbital integrals of \( O^* (x), O^* (x') \) equal

Problem 3 Extend to degenerate \( \mathcal{X} \# strip(r) \):
Find finitely many motives that govern the fundamental lemma for all \( X \in G \)

* Problem 4 Prove the motivic fundamental lemma.
(Develop a conceptual understanding of the motives that arise.)
Conclusion

The Denef-Loeser apparatus seems to mesh well with certain p-adic integrals that arise in representation theory.

We should investigate how far motives permeate representation theory of p-adic groups.

This should allow us to "calculate" p-adic integrals that we have found hard to calculate.