Schubert polynomials via triangulations of flow polytopes

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Based on joint work with Laura Escobar, Alex Fink and Avery St. Dizier.

Slide credits: Avery St. Dizier.
$\mathcal{R}_4 := $ ConvHull$(0, e_i - e_j \mid 1 \leq i < j \leq 4)$
Triangulating Root Polytopes
A pipe dream for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with crosses $\overline{\ }$ and elbows $\nearrow$ such that

- All tiles in the weak south-east triangle are elbows, and
- If you write 1, 2, \ldots, $n$ on the top and follow the strands (ignoring second crossings among the same strands), they come out on the left and read $\pi$ from top to bottom.

A pipe dream is **reduced** if no two strands cross twice.

![Figure: A reduced pipe dream for $\pi = 2143$.]
A Pipe Dream of 1432
Pipe Dreams to Noncrossing Alternating Spanning Trees
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Pipe Dreams to Noncrossing Alternating Spanning Trees
All Pipe Dreams of 1432
Triangulating Root Polytopes
Theorem (M. 2015)

For $\pi = 1n(n-1) \cdots 2$, the pipe dream complex of $\pi$ can be geometrically realized as a triangulation of the root polytope $\mathcal{R}_n := \text{ConvHull}(0, e_i - e_j \mid 1 \leq i < j \leq n)$. ★
Pipe dream complex for $\pi = 1432$

Thanks to Allen Knutson for the picture!
Theorem (M. 2015)

For \( \pi = 1n(n-1) \cdots 2 \), the pipe dream complex of \( \pi \) can be geometrically realized as a triangulation of the root polytope \( R_n := \text{ConvHull}(0, e_i - e_j \mid 1 \leq i < j \leq n) \).

In particular, the normalized volume of \( R_n \) equals the number of reduced pipe dreams of \( 1n(n-1) \cdots 2 \).
Theorem (M. 2015)

For $\pi = 1n(n-1) \cdots 2$, the pipe dream complex of $\pi$ can be geometrically realized as a triangulation of the root polytope $\mathcal{R}_n := \text{ConvHull}(0, e_i - e_j \mid 1 \leq i < j \leq n)$. *

In particular, the normalized volume of $\mathcal{R}_n$ equals the number of reduced pipe dreams of $1n(n-1) \cdots 2$.

We will see later that other root polytopes also have volumes expressed as the number of reduced pipe dreams of certain permutations.
Flow Polytopes

have subdivisions that give rise to

Right-Degree Polynomials

which are sometimes

Schubert Polynomials

motivating questions about

Newton Polytopes

Saturation

Grothendieck Polynomials
We will mostly be considering flow polytopes with a single source and sink. Start with a graph $G$. 

![Graph with a single source and sink]
We will mostly be considering flow polytopes with a single source and sink. Start with a graph $G$.

Fix an acyclic orientation of $G$. 

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,0);
\fill[black] (0,0) circle (3pt);
\fill[black] (1,0) circle (3pt);
\end{tikzpicture}
\end{center}
We will mostly be considering flow polytopes with a single source and sink. Start with a graph $G$.

\begin{center}
\begin{tikzpicture}
\node[circle, fill, inner sep=2pt] (v1) at (0,0) {};
\node[circle, fill, inner sep=2pt] (v2) at (2,0) {};
\node[circle, fill, inner sep=2pt] (v3) at (4,0) {};
\draw[->] (v1) -- (v2);
\draw[->] (v2) -- (v3);
\end{tikzpicture}
\end{center}

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\node[circle, fill, inner sep=2pt] (v3) at (4,0) {};
\draw[<->] (v1) -- (v2);
\draw[<->] (v2) -- (v3);
\end{tikzpicture}
\end{center}

*In this talk edges drawn without orientations should be assumed to be oriented left to right.*
Add a source $s$ and a sink $t$ connected to all the original vertices of $G$. Call the new graph $\tilde{G}$.
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Assign the source $s$ netflow 1, the sink $t$ netflow $-1$, and all other vertices netflow 0.
A **flow** on $\tilde{G}$ is an assignment of nonnegative real numbers to each edge of $\tilde{G}$ so that at every vertex, outflow minus inflow equals netflow.

The **flow polytope** $\mathcal{F}_{\tilde{G}}$ is the set of all flows on $\tilde{G}$ in $\mathbb{R}^{|E(\tilde{G})|}$.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & \frac{7}{12} & \frac{1}{4} \\
\frac{1}{2} & \frac{4}{3} & \frac{4}{3} & \frac{3}{6} & 0 & \frac{12}{12} & \frac{1}{4}
\end{pmatrix} \in \mathcal{F}_{\tilde{G}}$$
An Example Flow Polytope

$G$

$\tilde{G}$

$\mathcal{F}_{\tilde{G}}$
More generally, one can define $\mathcal{F}_G(\alpha_1, \ldots, \alpha_n)$ with no tilde and with any $\alpha_i$ summing to zero.

If $M_G$ is the incidence matrix of $G$, then

$$\mathcal{F}_G(\alpha_1, \ldots, \alpha_n) = \left\{ x \in \mathbb{R}_{\geq 0}^{E(G)} : M_G x = (\alpha_1, \alpha_2, \ldots, \alpha_n) \right\}$$

Consequently, the number of integer points in $\mathcal{F}_G$ is the number of ways to write $\alpha$ as a nonnegative linear combination of $\{e_i - e_j : (i, j) \in G\}$.

This vector partition function is the Kostant partition function $K_G(\alpha)$. 
Theorem (Postnikov-Stanley)

If $G$ is a graph on the vertex set $[n + 1],$

$$\text{Vol } \mathcal{F}_G(1, 0, \ldots, 0, -1) = K_G \left(0, d_2, \ldots, d_n, -\sum_{i=2}^{n} d_i\right)$$

where $d_i = \text{indeg}_G(i) - 1$ for each vertex $i.$

Consequently, the volume of any flow polytope with netflow $(1, 0, \ldots, 0, -1)$ is the number of integer points of another flow polytope.
Subdividing Flow Polytopes

Flow polytopes can be subdivided combinatorially by performing a sequence of changes to the original graph.

A reduction on a graph $G$ is a construction of two new graphs $G_1$ and $G_2$ from a choice of two adjacent edges $(i, j), (j, k) \in G$: 

\[
G \quad \xrightarrow{\quad} \quad G_1 \quad \xrightarrow{\quad} \quad G_2
\]
Subdividing Flow Polytopes

\[ G = P_4 \]

\[ \mathcal{F}_G \]

*This is an affine projection of \( \mathcal{F}_G \) onto the root polytope of \( G \).*
Subdividing Flow Polytopes
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Subdividing Flow Polytopes
More compactly, this subdivision procedure can be represented by a **reduction tree**.
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![Reduction Tree Diagram]
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The individual graphs appearing in a reduction tree depend on the choice of cuts used to dissect the flow polytope.
Are there subdivision invariants?

On the one hand, we have seen the leaves of a reduction tree are dependent on choices made.

On the other hand, the simplices produced by the reduction process are always unimodular, so the number of leaves in any reduction tree is always the normalized volume of the flow polytope regardless of any choices.

Question

Is there any stronger invariant across all the different ways to fully subdivide a flow polytope using reductions?
Is there an invariant of different subdivisions of a flow polytope?
Subdivisions to Degree Sequences
Is \( \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\} \) dependent only on the original graph?
Is \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\} dependent only on the original graph?

**Theorem (Grinberg 2017, M.-St. Dizier 2017)**

*Yes!*
Definition

For a graph $G$, let $RD(G)$ denote the multiset of right-degree sequences of the leaves in any reduction tree of $G$.

$RD(G) = \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\}$. 

\begin{align*}
\text{3} & \quad 0 & \quad 0 \\
\text{2} & \quad 0 & \quad 1 \\
\text{2} & \quad 1 & \quad 0 \\
\text{1} & \quad 2 & \quad 0 \\
\text{1} & \quad 1 & \quad 1
\end{align*}
Theorem (M.-St. Dizier 2017)

If $G$ is a graph on $[n + 1]$, $RD(G)$ is exactly the multiset of vectors of flow values on the edges $\{(s, i)\}_{i=1}^{n}$ in the flow polytope $\mathcal{F}_{\tilde{G}}(\#E(G), -d_1, -d_2, \ldots, -d_n, 0, 0)$, where $d_i = \text{outdeg}_G(i)$. 
Theorem (M.-St. Dizier 2017)

If $G$ is a graph on $[n+1]$, then the set $RD(G)$ is exactly the set of integer points of a projection of $\mathcal{F}_{\tilde{G}}(\#E(G), -d_1, \ldots, -d_n, 0, 0)$. 
Theorem (M.-St. Dizier 2017)

If $G$ is a graph on $[n + 1]$, then the set $RD(G)$ is exactly the set of integer points of a projection of $\mathcal{F}_{\tilde{G}}(\#E(G), -d_1, \ldots, -d_n, 0, 0)$. Moreover, the polytope we get by the aforementioned projection of $\mathcal{F}_{\tilde{G}}(\#E(G), -d_1, -d_2, \ldots, -d_n, 0, 0)$ is a generalized permutahedron.
The standard \textit{permutahedron} in $\mathbb{R}^n$ is the convex hull of all rearrangements of the vector $(1, 2, \ldots, n)$.

\textbf{Definition (Postnikov 2005)}

A \textit{generalized permutahedron} is any polytope obtained by deforming the standard permutahedron by moving the vertices in any way so that all edge directions are preserved.
The standard **permutahedron** in $\mathbb{R}^n$ is the convex hull of all rearrangements of the vector $(1, 2, \ldots, n)$.

**Definition/Theorem (Postnikov-Reiner-Williams 2006)**

A polytope $P \subset \mathbb{R}^n$ is a **generalized permutahedron** if and only if its reduced normal fan is refined by the braid arrangement fan.
Combinatorial/Geometric Interpretation

**Theorem (M.-St. Dizier 2017)**

If $G$ is a graph on $[n+1]$, $RD(G)$ is exactly the multiset of vectors of flow values on the edges $\{(s, i)\}_{i=1}^n$ in the flow polytope $\mathcal{F}_{\tilde{G}}(\#E(G), -d_1, -d_2, \ldots, -d_n, 0, 0)$, where $d_i = \text{outdeg}_G(i)$. 

![Diagram](image-url)
Define the **right-degree polynomial** of $G$ by

$$R_G(x) = \sum_{\alpha \in RD(G)} x^\alpha.$$ 

$RD(P_4) = \{(3, 0, 0), (2, 0, 1), (2, 1, 0), (1, 2, 0), (1, 1, 1)\}$

$$R_{P_4} = x_1^3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3$$
Schubert Polynomials

Geometrically, Schubert polynomials arise as distinguished representatives of the cohomology classes in the flag variety of \( \mathbb{C}^n \).

Combinatorially, Schubert polynomials can be described using divided difference operators \( \partial_i \):

\[
\partial_i f = \frac{f(x_1, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}
\]

If \( \pi_0 \) denotes the longest permutation \( n (n-1) \cdots 2 1 \) and \( s_i \) the transposition swapping \( i \) and \( i + 1 \),

\[
\mathcal{S}_\pi = \begin{cases} 
  x_1^{n-1}x_2^{n-2} \cdots x_{n-1} & \text{if } \pi = \pi_0 \\
  \partial_i \mathcal{S}_{\pi s_i} & \text{if } \pi \neq \pi_0, \pi(i) < \pi(i + 1)
\end{cases}
\]
Divided Difference Operators

\[ \partial_1(x_1^2x_2) = \frac{x_1^2x_2 - x_1x_2^2}{x_1 - x_2} = x_1x_2 \]

\[ \partial_2(x_1x_2) = \frac{x_1x_2 - x_1x_3}{x_2 - x_3} = x_1 \]

\[ \partial_1(x_1) = \frac{x_1 - x_2}{x_1 - x_2} = 1 \]

\[ \mathfrak{S}_\pi = \partial_{i_1} \cdots \partial_{i_k}(\mathfrak{S}_{\pi_0}) \]

for any reduced expression

\[ \pi^{-1}\pi_0 = s_{i_1} \cdots s_{i_k} \]
Theorem (Lascoux-Schützenberger 1982)

*The Schubert polynomial has nonnegative integer coefficients.*
Theorem (Billey-Jockusch-Stanley 1993)

The Schubert polynomial $S_w$ can be written as

$$S_w(x_1, \ldots, x_{n-1}) = \sum_{a \in R(w)} \sum_{(i_1, \ldots, i_p) \in K(a)} x_{i_1} \cdots x_{i_p},$$

where $a = (a_1, \ldots, a_p)$ is a reduced word for $w$, and $K(a)$ is the set of $a$-compatible sequences.
The Schubert polynomial $S_w$ can be written as

$$S_w(x_1, \ldots, x_{n-1}) = \sum_{a \in R(w)} \sum_{(i_1, \ldots, i_p) \in K(a)} x_{i_1} \cdots x_{i_p},$$

where $a = (a_1, \ldots, a_p)$ is a reduced word for $w$, and $K(a)$ is the set of $a$-compatible sequences.

A sequence $(i_1, \ldots, i_p) \in \mathbb{Z}_>^p$ is $a$-compatible if

- $i_1 \leq \cdots \leq i_p$
- $i_j \leq a_j$, $j \in [p]$
- $i_j < i_{j+1}$ if $a_j < a_{j+1}$. 

Theorem (Billey-Jockusch-Stanley 1993)
The monomials of the Schubert polynomial are given by reduced pipe dreams:

\[ S_w(x_1, \ldots, x_{n-1}) = \sum_P x^P \]

where \( x^P = \prod_{i=1}^{n-1} x_i \) \# crosses in row \( i \) of \( P \).
Pipe Dreams

A pipe dream for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with crosses $\rule{1cm}{0.5mm}$ and elbows $\rule{1cm}{0.5mm}$ such that

- All tiles in the weak south-east triangle are elbows, and
- If you write $1, 2, \ldots, n$ on the top and follow the strands (ignoring second crossings among the same strands), they come out on the left and read $\pi$ from top to bottom.

A pipe dream is reduced if no two strands cross twice.

![Figure: A reduced pipe dream for $\pi = 2143$.]
Pipe dreams and ladder moves

Theorem (Bergeron-Billey 1993)

All reduced pipe dreams of a permutation $\pi \in S_n$ can be obtained from the bottom pipe dream via ladder moves.
The Schubert Polynomial of 1432 is given by:

\[ S_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_2 \]
Calculating Right-Degree Polynomials

\[ P_4 = x_1^3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 \]
Pipe Dreams of 1432 to Trees
Theorem (M. 2015)

For the permutation $\pi = 1n(n - 1) \cdots 2$ the path $P_n$ on $n$ vertices has right-degree polynomial $R_{P_n}$ that is a reparameterization of $S_\pi$.

$$P_4$$

$$S_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$$

$$R_{P_4} = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3$$

$$S_{1432}(x_1, x_2, x_3) = x_1^3x_2^2x_3 R_{P_4}(x_1^{-1}, x_2^{-1}, x_3^{-1})$$
Theorem (Escobar-M. 2015)

For permutations of the form $\pi = 1\pi'$, where $\pi'$ is 132-avoiding, there is a tree $T_\pi$ such that the right-degree polynomial $R_{T_\pi}$ is a reparameterization of $S_\pi$.

\[
T_{1432}
\]

\[
S_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_2
\]

\[
R_{T_{1432}} = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3
\]

\[
S_{1432}(x_1, x_2, x_3) = x_1^3 x_2^2 x_3 R_{T_{1432}}(x_1^{-1}, x_2^{-1}, x_3^{-1})
\]
Definition (Knutson-Miller)

The **pipe dream complex** $PD(\pi)$ of a permutation $\pi \in S_n$ is the simplicial complex with vertices given by entries on the northwest triangle of an $n \times n$-matrix and facets given by the elbows in the reduced pipe dreams for $\pi$.
Pipe dream complex for $\pi = 1432$

Thanks to Allen Knutson for the picture!
Facts about pipe dream complexes

- Facets are labeled by reduced pipe dreams of $\pi$.
- Interior faces are labeled by pipe dreams of $\pi$.
- Boundary faces are labeled by pipe dreams of $w > \pi$ in Bruhat order.
Theorem (Knutson-Miller, 2004)

Pipe dream complexes are homeomorphic to balls for all permutations \( \pi \in S_n \) except for \( \pi = n(n-1) \cdots 1 \), for which they are homeomorphic to spheres.
Topology of Pipe Dream Complexes

Theorem (Knutson-Miller, 2004)

Pipe dream complexes are homeomorphic to balls for all permutations \( \pi \in S_n \) except for \( \pi = n(n-1) \cdots 1 \), for which they are homeomorphic to spheres.

Question

Can the pipe dream complex \( PD(\pi) \) be realized by a triangulation of a polytope?
### Theorem (Knutson-Miller, 2004)

Pipe dream complexes are homeomorphic to balls for all permutations \( \pi \in S_n \) except for \( \pi = n(n-1) \cdots 1 \), for which they are homeomorphic to spheres.

### Question

Can the pipe dream complex \( PD(\pi) \) be realized by a triangulation of a polytope?

### Theorem (Escobar-M. 2015)

For permutations of the form \( \pi = 1\pi' \) where \( \pi' \) is dominant (132-avoiding), there is a tree \( T_\pi \) such that the pipe dream complex of \( \pi \) can be geometrically realized as a triangulation of the root/flow polytope corresponding to \( T_\pi \). *
Theorem (Escobar-M. 2015)

For permutations of the form $\pi = 1\pi'$ where $\pi'$ is dominant (132-avoiding), there is a tree $T_\pi$ such that the pipe dream complex of $\pi$ can be geometrically realized as a triangulation of the root/flow polytope corresponding to $T_\pi$. ★

Root polytope $\mathcal{R}_T = \text{ConvHull}(0, e_i - e_j \mid (i < j) \in E(\bar{T}))$.

The flow polytope $\mathcal{F}_{\bar{T}}$ affinely projects to $\mathcal{R}_T$. 
Remember this reduction tree?
Noncrossing and Alternating Trees

A tree is **alternating** if it has no pair of edges

A tree is **noncrossing** if it has no pair of edges
Theorem (M. 2009)

Every noncrossing tree $T$ has a canonical reduction tree whose leaves are exactly the alternating noncrossing spanning trees of the directed transitive closure $\bar{T}$ of $T$. 

$T$ 

$\bar{T}$
Theorem (Escobar-M. 2015)

For permutations of the form $\pi = 1\pi'$ where $\pi'$ is dominant (132-avoiding), there is an acyclic graph $T_\pi$ such that the reduced pipe dreams of $\pi$ are in bijection with the noncrossing alternating spanning trees of the directed transitive closure $\overline{T_\pi}$. 
How do you construct $T_\pi$?
How do you construct $T_\pi$?
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How do you construct $T_{\pi}$?
Simplices in a subdivision of $\mathcal{F}_{\tilde{T}}$ by reductions

Leaves in any reduction tree of $T$

Leaves in the canonical reduction tree of $T$

Noncrossing alternating spanning trees of $\tilde{T}$

If $T = T_\pi$ for $\pi = 1\pi'$ with $\pi'$ dominant

Reduced pipe dreams of $\pi$
Pipe Dreams to Trees
Theorem (Escobar-M. 2015)

For permutations of the form $\pi = 1\pi'$ where $\pi'$ is dominant (132-avoiding), there is a tree $T_\pi$ such that the pipe dream complex of $\pi$ can be geometrically realized as a triangulation of the root/flow polytope corresponding to $T_\pi$. *

Root polytope $\mathcal{R}_\mathcal{T} = \text{ConvHull}(0, e_i - e_j | (i < j) \in E(\bar{T}))$.

The flow polytope $\mathcal{F}_{\bar{T}}$ affinely projects to $\mathcal{R}_\mathcal{T}$. 
Pipe Dream Complexes as Triangulations

**Theorem (Escobar-M. 2015)**

For permutations of the form $\pi = 1\pi'$ where $\pi'$ is dominant (132-avoiding), there is a tree $T_\pi$ such that the pipe dream complex of $\pi$ can be geometrically realized as a triangulation of the root/flow polytope corresponding to $T_\pi$. ∗

Root polytope $R_T = \text{ConvHull}(0, e_i - e_j \mid (i < j) \in E(\bar{T}))$.

The flow polytope $F_{\bar{T}}$ affinely projects to $R_T$.

**Corollary (Escobar-M. 2015)**

For permutations of the form $\pi = 1\pi'$ where $\pi'$ is dominant, the normalized volume of $R_{T_\pi}$ and $F_{\bar{T}_\pi}$ equal the number of reduced pipe dreams of $\pi$. 
Right-Degree and Schubert Polynomials

Schubert Polynomials

$1\pi'$ Case

Right-Degree Polynomials

Similar Properties?
Roadmap

Flow Polytopes

Right-Degree Polynomials

Schubert Polynomials

Newton Polytopes
Saturation
Grothendieck Polynomials
Newton Polytopes

Any polynomial \( f = \sum_{z \in \mathbb{Z}^n} a_z x^z \in \mathbb{C}[x_1, \ldots, x_n] \) has an associated integer polytope called its Newton polytope:

\[
\text{Newton}(f) = \text{Conv}(z : a_z \neq 0)
\]

Newton \((1 + x + y + x^2 + xy^2 + x^2y^2) = \)

Newton \((1 + y + xy + xy^2 + x^2 + x^2y^2) = \)
Recall...

**Theorem (M.-St. Dizier 2017)**

If $G$ is a graph on $[n + 1]$, then the set $RD(G)$ is exactly the set of integer points of a projection of $\tilde{\mathcal{F}}_G(\#E(G), -d_1, \ldots, -d_n, 0, 0)$. Moreover, the polytope we get by the aforementioned projection of $\tilde{\mathcal{F}}_G(\#E(G), -d_1, -d_2, \ldots, -d_n, 0, 0)$ is a generalized permutahedron.
An Answer For $R_G$

**Theorem (M.-St. Dizier 2017)**

... For any graph $G$, $\text{Newton}(R_G)$ is a generalized permutahedron.

$$T_{1432}$$

$$\text{Newton}(R_{T_{1432}}) =$$

$$(1, 1, 1), (2, 1, 0), (2, 0, 1), (3, 0, 0)$$
Theorem (M.-St. Dizier 2017)

... For any graph $G$, $\text{Newton}(R_G)$ is a generalized permutahedron.

Question

Is $\text{Newton}(G_\pi)$ a generalized permutahedron for all $\pi \in S_n$?
What kind of polytopes are the Newton polytopes of Schubert polynomials?

\[ \mathcal{S}_{1243} = x_1 + x_2 + x_3 \]

\[ \mathcal{S}_{13524} = x_2 x_3^2 + x_1 x_3^2 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + 2 x_1 x_2 x_3 \]
\[ S_{21543} = x_1^3 x_2 + x_1^3 x_3 + x_1^3 x_4 + x_1^2 x_2^2 + x_1^2 x_3^2 + 2 x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1 x_3 x_4 + x_1 x_2 x_3^2 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_1 x_2 x_3 x_4 \]
Conjecture (Monical-Tokcan-Yong 2017)

For any $\pi \in S_n$, $\text{Newton}(S_\pi)$ is a generalized permutahedron.
Schubert Newton Polytopes

Conjecture (Monical-Tokcan-Yong 2017)

For any $\pi \in S_n$, $\text{Newton}(\mathcal{G}_\pi)$ is a generalized permutahedron.

- $\{\text{Schubert polynomials } \mathcal{G}_\pi\} \supseteq \{\text{Schur polynomials } s_\lambda\}$
Conjecture (Monical-Tokcan-Yong 2017)

For any $\pi \in S_n$, $\text{Newton}(\mathcal{S}_\pi)$ is a generalized permutahedron.

- $\{\text{Schubert polynomials } \mathcal{S}_\pi\} \supseteq \{\text{Schur polynomials } s_\lambda\}$
- $\text{Newton}(s_\lambda) = \text{Conv} (\text{all permutations of } \lambda) = \text{permutahedron}$
Schubert Newton Polytopes

Conjecture (Monical-Tokcan-Yong 2017)

For any $\pi \in S_n$, $\text{Newton}(\mathcal{S}_\pi)$ is a generalized permutahedron.

- \{Schubert polynomials $\mathcal{S}_\pi$\} $\supseteq$ \{Schur polynomials $s_\lambda$\}
- $\text{Newton}(s_\lambda) = \text{Conv}(\text{all permutations of } \lambda) =$ permutahedron
- $\text{Newton}(\mathcal{S}_\pi)$ should be a generalized permutahedron
Conjecture (Monical-Tokcan-Yong 2017)

For any $\pi \in S_n$, $\text{Newton}(S_\pi)$ is a generalized permutahedron.

- $\{\text{Schubert polynomials } S_\pi\} \supseteq \{\text{Schur polynomials } s_\lambda\}$
- $\text{Newton}(s_\lambda) = \text{Conv}(\text{all permutations of } \lambda) = \text{permutahedron}$
- $\text{Newton}(S_\pi)$ should be a generalized permutahedron

Theorem (Fink-M.-St. Dizier 2017)

The Newton polytopes of the Schubert polynomials are generalized permutahedra.
Theorem (Fink-M.-St. Dizier 2017)

The Newton polytopes of the Schubert polynomials are generalized permutahedra.

To prove this, we associate a matroid $M_i$ to each column of the Rothe diagram of $\pi$ and show that the Newton polytope decomposes as the Minkowski sum

$$\text{Newton}(\mathcal{G}_\pi) = P(M_1) + \cdots + P(M_n)$$

of the matroid polytopes.

From this we also extract an inequality description of the Schubert polynomial Newton polytope conjectured by Monical-Tokcan-Yong.
A Saturation Property of $R_G$

What does $RD(G)$ look like? Specifically, how do the points in $RD(G)$ sit inside the Newton polytope of $R_G$?

$T_{1432} = (1, 1, 1) \rightarrow (2, 1, 0) \rightarrow (3, 0, 0) \rightarrow (2, 0, 1) \rightarrow (1, 2, 0)$

Newton($RT_{1432}$) =

$T_{1432}$

(1, 2, 0)

(1, 1, 1)

(2, 1, 0)

(2, 0, 1)

(3, 0, 0)
What does $RD(G)$ look like? Specifically, how do the points in $RD(G)$ sit inside the Newton polytope of $R_G$?

Newton($R_{T_{1432}}$) =

\begin{align*}
(1, 1, 1) & \\
(2, 0, 1) & \\
(3, 0, 0) & \\
(2, 1, 0) & \\
(1, 2, 0) & \\
\end{align*}

Theorem (M.-St. Dizier 2017)

Newton($R_G$) is a generalized permutahedron whose integral points are exactly $RD(G)$.
Saturated Newton Polytopes

Definition (Monical-Tokcan-Yong 2017)

A polynomial \( f \) is said to have **saturated Newton polytope** (SNP) if every integer point in the Newton polytope corresponds to a monomial with nonzero coefficient in \( f \).

\[
\mathcal{S}_{13524} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3
\]
Theorem (Monical-Tokcan-Yong 2017)

The following all have SNP:

- Schur polynomials
- Skew-Schur polynomials
- Stanley symmetric function
- \((q, t)\) evaluations of symmetric Macdonald polynomials
### Theorem (Monical-Tokcan-Yong 2017)

*The following all have SNP:*

- *Schur polynomials*
- *Skew-Schur polynomials*
- *Stanley symmetric function*
- *(q, t) evaluations of symmetric Macdonald polynomials*

### Conjecture (Monical-Tokcan-Yong 2017)

*The following all have SNP:*

- *Schubert polynomials*
- *Key polynomials*
- *Double Schubert polynomials*
- *Grothendieck polynomials*
Theorem

The following all have SNP:

- Schubert polynomials (Fink-M.-St. Dizier 2017)
- Key polynomials (Fink-M.-St. Dizier 2017)
- $1\pi'$ Grothendieck polynomials (M.-St. Dizier 2017)
- Symmetric Grothendieck polynomials (Escobar-Yong 2017)
Instead of constructing two new graphs $G_1$ and $G_2$ when performing a reduction on two adjacent edges $(i, j), (j, k) \in G$ construct three:
Full Reduction Tree
Right-Degree Sequences

**Definition**
For a graph $G$, let $FRD(G)$ denote the multiset of right-degree sequences of the leaves in any reduction tree of $G$.

$FRD(G) = \{(3, 1, 0), (2, 1, 0), (2, 2, 0), (2, 1, 0), (3, 1, 0), (2, 1, 1), (2, 1, 0)\}$. 
Define the **full right-degree polynomial** of $G$ by

$$R_G^f(x) = \sum_{\alpha \in \text{FRD}(G)} (-1)^{\#E(G) - |\alpha|} x^\alpha.$$ 

$$\text{FRD}(G) = \{(3,1,0), (2,1,0), (2,2,0), (2,1,0), (3,1,0), (2,1,1), (2,1,0)\}$$

\[ R_G^f = x_1^3x_2 - x_1^2x_2 + x_1^2x_2^2 - x_1^2x_2 \]
The Construction of Grothendieck Polynomials

Geometrically, Grothendieck polynomials arise as distinguished representatives of the K-theory classes in the flag variety of $\mathbb{C}^n$.

**Theorem (Fomin-Kirillov 1994)**

The monomials of the Grothendieck polynomial $G_\pi$ are given by all pipe dreams of $\pi$:

$$G_\pi(x_1, \ldots, x_{n-1}) = (-1)^{l(\pi) - \# \text{ crosses in } P} \sum_P x^P$$

where $x^P = \prod_{i=1}^{n-1} x_i^{\# \text{ crosses in row } i \text{ of } P}$. 
Theorem (Escobar-M. 2015)

For permutations of the form $\pi = 1\pi'$, where $\pi'$ is 132-avoiding, there is a tree $T_\pi$ such that the full right-degree polynomial $R^f_{T_\pi}$ is a reparameterization of $\mathcal{G}_\pi$.

$T_{1432}$

$\mathcal{G}_{1432} = x_2^2 x_3 - x_2^2 x_1^2 + x_2 x_1^2 + x_2^2 x_3 x_1^2 - 2 x_2 x_3 x_1^2 + x_3 x_1^2 + x_2^2 x_1 - 2 x_2^2 x_3 x_1 + x_2 x_3 x_1$

$R^f_{T_{1432}} = x_1^3 + x_2 x_1^2 + x_3 x_1^2 - 2 x_1^2 + x_2^2 x_1 - 2 x_2 x_1 + x_2 x_3 x_1 - x_3 x_1 + x_1$

$\mathcal{G}_{1432}(x_1, x_2, x_3) = x_1^3 x_2^2 x_3 R^f_{T_{1432}}(x_1^{-1}, x_2^{-1}, x_3^{-1})$
One way to compute this is...

\[ x_{ij} x_{jk} \rightarrow x_{ik} x_{ij} + x_{ik} x_{jk} + \beta x_{ik} \]

for \( i < j < k \)
Full Reduction Tree
Computing $R_G$

\[ x_{13}^2 x_{23} x_{34} \]
Starting with the monomial $x_{13}^2 x_{23} x_{34}$, repeatedly apply

$$x_{ij} x_{jk} \rightarrow x_{ik} x_{ij} + x_{ik} x_{jk} + \beta x_{ik} \text{ for } i < j < k.$$ 

Obtain a polynomial in $x_{ij}$ that is a sum over the leaves of the reduction tree.

Let $x_{ij} \leftrightarrow x_i$. 

Computing $R_G$
Computing $R_G$
Computing $R_G$
Computing $R_G$
Right degree polynomial \( R_G \)
Another way to compute $R_G$

**Theorem (M.-St. Dizier 2017)**

If $G$ is a graph on $[n + 1]$, $FRD(G)$ is exactly the multiset of vectors of flow values on certain edges in a capacitated flow polytope.

$$FRD(T_{1432}) = \{(a_1, a_2, a_3)\}$$
Another way to compute $R_G$

**Theorem (M.-St. Dizier 2017)**

*If $G$ is a graph on $[n + 1]$, $FRD(G)$ is exactly the multiset of vectors of flow values on certain edges in a capacitated flow polytope.*

Newton polytope, saturation...