Reflections on cylindrical contact homology

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**Contact structures**

**Definition**

A **contact structure** is a maximally nonintegrable hyperplane field.

\[ \xi = \ker(dz - ydx) \]

The kernel of a 1-form \( \lambda \) on \( Y^{2n-1} \) is a contact structure whenever

- \( \lambda \wedge (d\lambda)^{n-1} \) is a volume form \( \Leftrightarrow d\lambda|_\xi \) is nondegenerate.
Reeb vector fields

**Definition**

The Reeb vector field \( R \) on \((Y, \lambda)\) is uniquely determined by

- \( \lambda(R) = 1 \),
- \( d\lambda(R, \cdot) = 0 \).

The **Reeb flow**, \( \varphi_t : Y \rightarrow Y \) is defined by \( \frac{d}{dt} \varphi_t(x) = R(\varphi_t(x)) \).

A closed **Reeb orbit** (modulo reparametrization) satisfies

\[ \gamma : \mathbb{R} / T\mathbb{Z} \rightarrow Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \quad (0.1) \]

and is **embedded** whenever \((0.1)\) is injective.
Reeb orbits

Given an embedded Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$, the linearized flow along $\gamma$ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

d$\varphi_T$ is called the linearized return map.

If 1 is not an eigenvalue of $d\varphi_T$ then $\gamma$ is nondegenerate.

$\lambda$ is nondegenerate if all Reeb orbits associated to $\lambda$ are nondegenerate.

In dim 3, nondegenerate orbits are either elliptic or hyperbolic according to whether $d\varphi_T$ has eigenvalues on $S^1$ or real eigenvalues.
Reeb orbits on $S^3$

$S^3 := \{(u, v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}$, $\lambda = \frac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv)$.

The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu \frac{\partial}{\partial u} - i\bar{u} \frac{\partial}{\partial \bar{u}} + iv \frac{\partial}{\partial v} - i\bar{v} \frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u, v) = (e^{it}u, e^{it}v)$. 

Patrick Massot

Niles Johnson, $S^3/S^1 = S^2$
The Hopf Fibration

Niles Johnson
http://www.nilesjohnson.net
Helmut Hofer on the origins of the field:

*So why did I come into symplectic and contact geometry? So it turned out I had the flu and the only thing to read was a copy of Rabinowitz’s paper where he proves the existence of periodic orbits on star-shaped energy surfaces. It turned out to contain a fundamental new idea, which was to study a different action functional for loops in the phase space rather than for Lagrangians in the configuration space. Which actually if we look back, led to the variational approach in symplectic and contact topology, which is reincarnated in infinite dimensions in Floer theory and has appeared in every other subsequent approach. ...Ja, the flu turned out to be really good.*
Existence of periodic orbits

The Weinstein Conjecture (1978)

Let $Y$ be a closed oriented odd-dimensional manifold with a contact form $\lambda$. Then the associated Reeb vector field $R$ admits a closed orbit.

- Weinstein (convex hypersurfaces)
- Rabinowitz (star shaped hypersurfaces)
- Star shaped is secretly contact!
- Viterbo, Hofer, Floer, Zehnder (‘80’s fun)
- Hofer ($S^3$)
- Taubes (dimension 3)

Let \((Y^{2n-1}, \xi = \ker \lambda)\) be a closed nondegenerate contact manifold.

Floerify Morse theory on

\[ \mathcal{A} : C^\infty(S^1, Y) \to \mathbb{R}, \quad \gamma \mapsto \int_\gamma \lambda. \]

Proposition

\(\gamma \in \text{Crit}(\mathcal{A}) \iff \gamma \text{ is a closed Reeb orbit.} \)

- Grading: \(|\gamma| = CZ(\gamma) + n - 3,\)
- \(C_{EGH}^*(Y, \lambda, J) = \mathbb{Q}\langle\{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}\rangle\)
- 3-D: Even covers of embedded negative hyperbolic orbits are bad.
The letter $J$ is for pseudoholomorphic

A $\lambda$-compatible almost complex structure is a $J$ on $T(\mathbb{R} \times Y)$:

- $J$ is $\mathbb{R}$-invariant
- $J\xi = \xi$, positively with respect to $d\lambda$
- $J(\partial_s) = R$, where $s$ denotes the $\mathbb{R}$ coordinate

Gradient flow lines are a no go; instead count **pseudoholomorphic cylinders** $u \in \mathcal{M}^J(\gamma_+, \gamma_-)/\mathbb{R}$.

$$u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times Y, J)$$

$$\bar{\partial}_J u := du + J \circ du \circ j \equiv 0$$

$$\lim_{s \to \pm \infty} \pi_{\mathbb{R}} u(s, t) = \pm \infty$$

$$\lim_{s \to \pm \infty} \pi_{\mathcal{M}} u(s, t) = \gamma_{\pm}$$

**Note:** $J$ is $S^1$-INDEPENDENT

up to reparametrization.
Cylindrical contact homology

- $C_*(Y, \lambda, J) = \mathbb{Q}\langle \{\text{closed} \} \setminus \{\text{bad} \} \rangle$
- $|\gamma| = CZ(\gamma) + n - 3$
- $\langle \partial^{EGH} \alpha, \beta \rangle = \sum_{u \in \mathcal{M}^J(\alpha, \beta)/\mathbb{R}, \ |\alpha| - |\beta| = 1} \frac{m(\alpha)}{m(u)} \epsilon(u)$
- $\langle \partial^{H\Phi} \alpha, \beta \rangle = \sum_{u \in \mathcal{M}^J(\alpha, \beta)/\mathbb{R}, \ |\alpha| - |\beta| = 1} \frac{m(\beta)}{m(u)} \epsilon(u)$

Conjecture (Eliashberg-Givental-Hofer ‘00)

*If there are no contractible Reeb orbits with $|\gamma| = -1, 0, 1$ then $(C_*, \partial)$ is a chain complex and $CH^{EGH}_*(Y, \ker \lambda; \mathbb{Q}) = H(C_*(Y, \lambda, J), \partial)$ is an invariant of $\xi = \ker \lambda$.\*
**Definition**

- $(Y^{2n+1}, \lambda)$ is **hypertight** if there are no contractible Reeb orbits.
- $(Y^3, \lambda)$ is **dynamically convex** whenever
  
  $$c_1(\xi)|_{\pi_2(Y)} = 0$$
  
  and every contractible $\gamma$ satisfies $CZ(\gamma) \geq 3$.

For us $\{\text{hypertight}\} \subset \{\text{dynamically convex}\}$.

A convex hypersurface transverse to the radial vector field $Y$ in $(\mathbb{R}^4, \omega_0)$ admits a dynamically convex contact form $\lambda_0 := \omega_0(Y, \cdot)$.

**Theorem (Hutchings-N. ‘14 (JSG 2016))**

*If $(Y^3, \lambda)$ is dynamically convex, $J$ generic, and every contractible Reeb orbit $\gamma$ has $CZ(\gamma) = 3$ only if $\gamma$ is embedded then $\partial^2 = 0$.***

- Intersection theory is key to our proof that $\partial^2 = 0$.
- Can allow contractible $CZ(\gamma) = 3$ for prime covers of embedded Reeb orbits
  
  (Cristofaro-Gardiner - Hutchings - Zhang)

- 3D hypertight: invariance via obg for chain homotopy (Bao - Honda ‘14)
- Any dim hypertight: $\partial^2 = 0$ and invariance via Kuranishi atlases (Pardon ‘15)
The pseudoholomorphic menace

- Transversality for multiply covered curves is hard.
- Is $\mathcal{M}^J(\gamma_+;\gamma_-)$ more than a set?
- $\mathcal{M}^J(\gamma_+;\gamma_-)$ can have nonpositive virtual dimension...
- Compactness issues are “severe”.

Desired compactification when $\text{CZ}(x) - \text{CZ}(z) = 2$. 

Adding to 2 becomes hard.
The return of regularity (domain dependent $\mathcal{J}$)

- $S^1$-independent $J$ cylinders in $\mathbb{R} \times Y^3$ are reasonable
- All hope is lost in cobordisms, and no obvious chain maps.
- Invariance of $CH_{EGH}^\ast(Y, \lambda, J)$ requires $S^1$-dependent $\mathcal{J} := \{J_t\}_{t \in S^1}$.
- But breaking $S^1$-symmetry invalidates $(\partial^{EGH})^2 = 0$.
- We define a Morse-Bott non-equivariant chain complex

$$NCC_\ast := \bigoplus_{\text{all Reeb orbits } \gamma} \mathbb{Z} \langle \tilde{\gamma}, \hat{\gamma} \rangle, \quad \partial^{NCH} := \begin{pmatrix} \tilde{\partial} & \partial^+ \\ \partial^- & \text{obg} & \hat{\partial} \end{pmatrix}$$

- If sufficient regularity exists to use $J$, then between good orbits,
  $$\tilde{\partial} = \partial^{EGH}, \quad \hat{\partial} = -\partial^{H\mathcal{J}E}, \quad \partial^+ = 0$$
- Compactness issues require obstruction bundle gluing, producing a novel correction term.
- The nonequivariant theory $NCH_\ast$ is a contact invariant, which we relate to $CH_{EGH}^\ast$ via family Floer methods.
Enter the point constraints

Given a generic $\lambda$-compatible family $\mathbb{J} := \{J_t\}_{t \in S^1}$,

$$e_\pm : \mathcal{M}^\mathbb{J}(\gamma_+, \gamma_-) \rightarrow \text{im}(\gamma_\pm) = \overline{\gamma}_\pm$$

$$u \mapsto \lim_{s \rightarrow \pm \infty} \pi_\gamma u(s, 0)$$

Can use to specify a generic **base point** $p_{\overline{\gamma}}$ on each embedded $\overline{\gamma}$: $e_+(u) = p_{\alpha}, \ e_-(u) = p_{\beta}$.

The base level **cascade Morse-Bott moduli spaces**, $\mathcal{M}^\mathbb{J}(\cdot, \cdot)_1$:

$$\mathcal{M}^\mathbb{J} (\hat{\alpha}, \hat{\beta})_1 := \mathcal{M}^\mathbb{J}(\alpha, \beta)$$

$$\mathcal{M}^\mathbb{J} (\bar{\alpha}, \bar{\beta})_1 := \{ u \in \mathcal{M}^\mathbb{J}(\alpha, \beta) \mid e_+(u) = p_{\alpha} \}$$

$$\mathcal{M}^\mathbb{J} (\tilde{\alpha}, \tilde{\beta})_1 := \{ u \in \mathcal{M}^\mathbb{J}(\alpha, \beta) \mid e_-(u) = p_{\beta} \}$$

$$\mathcal{M}^\mathbb{J} (\check{\alpha}, \check{\beta})_1 := \{ u \in \mathcal{M}^\mathbb{J}(\alpha, \beta) \mid e_+(u) = p_{\alpha}, \ e_-(u) = p_{\beta} \}$$

Higher levels consist of certain tuples $(u_1, \ldots, u_\ell)$ of broken cylinders.

As a set, each of these spaces is a disjoint union of subsets $\mathcal{M}^\mathbb{J}(\cdot, \cdot)_\ell$. 
Enter the cascade moduli spaces

**Definition**

Assuming \( \alpha = \gamma_0, \gamma_1, \ldots, \gamma_{\ell-1}, \gamma_\ell = \beta \) are all distinct, the **higher levels** \( \mathcal{M}^J_{|\alpha|-|\beta|} (\tilde{\alpha}, \tilde{\beta})_{\ell} \), are the set of tuples

\[
(u_1, \ldots, u_\ell) \in \prod_{i=1}^{\ell} \mathcal{M}^J (\gamma_{i-1}, \gamma_i)
\]

such that

- if \( \tilde{\alpha} = \check{\alpha} \) then \( e_+ (u_0) = p_\check{\alpha} \);
- if \( \tilde{\beta} = \check{\beta} \) then \( e_- (u_\ell) = p_{\check{\beta}} \);
- \( e_- (u_{i-1}), e_+ (u_i), p_{\gamma_i} \), are cyclically ordered wrt Reeb flow.

When \( \alpha = \beta \), define

\[
\mathcal{M}^J (\check{\alpha}; \check{\alpha}) = \mathcal{M}^J (\check{\alpha}; \check{\alpha}) = \mathcal{M}^J (\check{\alpha}; \check{\alpha}) = \emptyset,
\]

\[
\mathcal{M}^J (\check{\alpha}; \check{\alpha}) := \begin{cases} 
-2 \{\text{pt}\} & \text{if } \alpha \text{ is bad;} \\
\emptyset & \text{if } \alpha \text{ is good.}
\end{cases}
\]
A new hope for a chain complex

\[ NCC_* := \bigoplus_{\text{all Reeb orbits } \gamma} \mathbb{Z}\langle \vec{\gamma}, \hat{\gamma} \rangle, \quad \partial^{NCH} := \begin{pmatrix} \bar{\partial} & \partial^+ \\ \partial^- + \text{obg} & \hat{\partial} \end{pmatrix} \]

\[ \tilde{\partial} : \quad \overline{CC}_* \to \overline{CC}_{*-1} \quad \partial^+ : \quad \overline{CC}_* \to \overline{CC}_* \]
\[ \tilde{\alpha} \mapsto \sum_{\begin{subarray}{c} \tilde{\beta}, |\alpha| - |\beta| = 1 \\ u \in \mathcal{M}^J(\tilde{\alpha}, \tilde{\beta}) \end{subarray}} \varepsilon(u)\tilde{\beta} \quad \hat{\alpha} \mapsto \sum_{\begin{subarray}{c} \tilde{\beta}, |\alpha| - |\beta| = 0 \\ u \in \mathcal{M}^J(\hat{\alpha}, \tilde{\beta}) \end{subarray}} \varepsilon(u)\tilde{\beta} \]

\[ \partial^- : \quad \overline{CC}_* \to \overline{CC}_{*-2} \quad \tilde{\partial} : \quad \overline{CC}_* \to \overline{CC}_{*-1} \]
\[ \tilde{\alpha} \mapsto \sum_{\begin{subarray}{c} \tilde{\beta}, |\alpha| - |\beta| = 2 \\ u \in \mathcal{M}^J(\tilde{\alpha}, \tilde{\beta}) \end{subarray}} \varepsilon(u)\tilde{\beta} \quad \hat{\alpha} \mapsto \sum_{\begin{subarray}{c} \tilde{\beta}, |\alpha| - |\beta| = 1 \\ u \in \mathcal{M}^J(\hat{\alpha}, \tilde{\beta}) \end{subarray}} \varepsilon(u)\tilde{\beta} \]

Theorem (Hutchings-N '19 (obg details in progress))

If \((Y^{2n-1}, \lambda)\) is hypertight or \((Y^3, \lambda)\) is dynamically convex, then for a generic family \(\mathcal{J}\), \(\partial^{NCH}\) is well-defined, \((\partial^{NCH})^2 = 0\), and \(NCH_*(Y, \ker \lambda)\) is independent of the choice of \(\lambda\) and \(\mathcal{J}\).
Rise of the obstruction bundles \((\partial^- + \text{obg})\tilde{\partial} + \hat{\partial}(\partial^- + \text{obg}) = 0\)

Given \(\Sigma_R \to \mathbb{R} \times S^1\) we can form a preglued curve. Next try to perturb to an honest pseudoholomorphic curve.

- Near \(x\) of \(\Sigma\) only perturb in directions normal to \(\mathbb{R} \times \gamma\)
- Obtain a unique curve iff the gluing obstruction \(s(\Sigma) = 0\), where \(s\) is a section of the obstruction bundle \(\mathcal{O} \to \mathcal{M}_R\).
- Count of gluings is related to count of 0's of obstruction section \(s\).
- Fiber = \(\text{coker}(D_\Sigma)^*\), Rank = \(\dim \mathcal{M}_R\).
- \([HT]\): branch points varied but objects being glued are fixed.
- \([HN]\): \(x\) is fixed but the glued object \(P\) varies in its moduli space.

**Theorem (Hutchings-N, in progress)**

Let \((Y^3, \lambda)\) be dynamically convex and \(J\) generic. If \(\gamma\) is an embedded elliptic contractible Reeb orbit then

\[
\langle (\text{obg})^{\gamma^d}, \gamma^{d-1} \rangle = n(\gamma),
\]

given by the leading coefficient of the asymptotic op associated to \(P\):

\[
n(\gamma) = \deg (\mathcal{M}^d(\text{index 2 planes asymptotic to } \gamma)/\mathbb{R} \to S^1),
\]
Essentialness of \textbf{obg} for the ellipsoid \( E(a,b) = \left\{ (u,v) \in \mathbb{C}^2 \mid \frac{|u|^2}{a} + \frac{|v|^2}{b} = 1 \right\} \)

Let \( \alpha \) and \( \beta \) be orbits for the ellipsoid with \( |\alpha| - |\beta| = 2 \).

\textbf{The differential coefficient from} \( \hat{\alpha} \to \tilde{\beta} \) \textbf{has to be} \( \pm 1 \) \textbf{or else the homology comes out wrong.}

The \textbf{obg} arises when \( \alpha = \gamma^{k+1}, \beta = \gamma^k \) with \( \gamma \) the short orbit.

\textbf{HWZ: the holomorphic planes bounded by} \( \gamma \) \textbf{give a foliation of} \( E(a,b) \).

It follows directly from this that the obstruction bundle term is \( \pm 1 \), assuming you know what you are doing (stay tuned).

An intersection theory argument shows there are no non-\textbf{obg} contributions to the differential coefficient from \( \hat{\alpha} \) to \( \tilde{\beta} \).
Obstruction bundle gluing setup

\[ \gamma \text{ is embedded, } C \text{ and } P \text{ are immersed} \]
\[ \Sigma \text{ is a branched cover of } \gamma \times \mathbb{R} \]
\[ \text{Fix } \mathbb{R}\text{-coor of } \mathbf{x} \text{ (akin to gluing parameter)} \]
\[ \dim (\text{Coker} \, D_{\Sigma}) = 2 \]
\[ \text{Fix point constraint at bottom of } \Sigma \]
\[ \text{Fix translation of } C \text{ (another gluing parameter)} \]

After fixing the \( \mathbb{R}\)-coor of \( C, \Sigma, P \) we have three degrees of freedom:

1. \( S^1\)-coordinate of the branch point \( \mathbf{x} \).
2. Choice of \( P \), a point in the moduli space of planes.

We have three constraints:

1. Conformal constraint corresponding to the point constraint.
2. The gluing obstruction.

\[ \exists ! \text{ choice of } S^1\text{-coor agreeing with the conformal constraint}. \]

Next, the gluing obstruction...
The gluing obstruction

- Given nonzero $\psi \in \text{coker}(D_\Sigma) = \text{ker}(D_\Sigma^*)$, consider its asymptotic eigenfunctions $\{\psi_i\}$. $\psi$ is a 1-D $\mathbb{C}$-vector space and hence a section.
- Let $\psi_d, \psi_1$ be in the leading eigenspace of the asymptotic operators $L_{\gamma_d}, L_\gamma$. (espace for $L_\gamma$ pulls back to espace for $L_{\gamma_d}$).
- Take $\psi_C, \psi_P$ to be the associated asymptotic eigenfunctions for Cylinder and Plane.
- If $J$ is generic then $\psi_d, \psi_1, \psi_C, \psi_P$ are all nonvanishing.

The gluing obstruction comes from a count of zeros.

Many pages of math permit use of the following approximation given by the coefficients of the leading order term

$$s(\Sigma) \approx \langle \psi_d, \psi_C \rangle + \langle \psi_1, \psi_P \rangle,$$

Everything is fixed except for $\psi_P$ as $P$ can move in its moduli space.

Since we fixed $\mathbf{x}$ for $\Sigma$, the number of ways to glue is given by the choices of $P$ such that $\langle \psi_1, \psi_P \rangle = -\langle \psi_d, \psi_C \rangle$

It suffices to find the zeros of the linearized section, given by the coefficients of the leading order terms: $s_0(\Sigma) = a_C \psi_C + a_P \psi_P$

$\exists$ unique $a_P \in \mathbb{C} \setminus \{0\}$ because $\psi$ is a 1-D $\mathbb{C}$-vector space $\cong \mathbb{C}$.

Translation in $\mathbb{R}$ corresponds to multiplication by $e^s$
OBSTRUCTION BUNDLE  ZOOMING

“It’s the best paper you’ve never read!”  –Ko Honda, 2020

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Join us weekly in September, someday after 10am PDT for the fun!
Family Floer (cf. Bourgeois-Oancea IMRN ’17)

\[ CC^S_1(Y, \lambda) = NCC_* \otimes \mathbb{Z}[U], \text{ deg}(U) = 2, \partial^S_1 = \partial^{NCH} \otimes 1 + \ldots + \partial_k \otimes U^{-k} + \ldots \]

- Let \( \mathcal{J} \) be an \( S^1 \)-equivariant \( S^1 \times ES^1 \) family.
- Fix a perfect Morse \( f \) on \( BS^1 = S^\infty \).
- Given \( \gamma_{\pm} \) and \( x_{\pm} \in \text{Crit}(f) \), consider pairs \((\eta, u)\) of grad flow \( \eta : \mathbb{R} \to ES^1 = \mathbb{C}P^\infty \) asymptotic to points in \( \pi^{-1}(x_{\pm}) \) and \( u : \mathbb{R} \times S^1 \to \mathbb{R} \times Y, \partial_s u + \mathcal{J}_{t, \eta(s)} u = 0 \), asymptotic to \( \gamma_{\pm} \).
- Let \( M^J((x_+, \gamma_+), (x_-, \gamma_-)) \) be the quotient of this solution set.
- Have evaluation maps and can run Axiomatic \( S^1 \) Morse-Bott framework (HN ’17).

Theorem (Hutchings-N ’19 (obg details in progress))

If \((Y^{2n-1}, \lambda)\) is hypertight or \((Y^3, \lambda)\) is dynamically convex, then for a generic family \( J \), \((CC^S_1(Y, \lambda, J), \partial^S_1)\) is a chain complex and \( CH^S_1(Y, \ker \lambda) \) is independent of the choice of \( \lambda \) and \( J \).
Suppose $J$ is $\lambda$-compatible on $\mathbb{R} \times Y$ which satisfies the necessary transversality conditions to define $\partial^{EGH}$ and show $(\partial^{EGH})^2 = 0$. Then we can use the “autonomous” family $\tilde{J} = \{J\}$:

$$\partial^{S^1} = \partial^{NCH} \otimes 1 + \partial_1 \otimes U^{-1},$$

where the “BV operator” $\partial_1$ is given by

$$\partial_1 \tilde{\alpha} = 0, \quad \partial_1 \check{\alpha} = \begin{cases} 
   d(\alpha) \hat{\alpha}, & \alpha \text{ good}, \\
   0, & \alpha \text{ bad}.
\end{cases}$$

If $\alpha$ and $\beta$ are good Reeb orbits, then

$$\langle \partial^{NCH} \tilde{\alpha}, \hat{\beta} \rangle = \langle \partial^{EGH} \alpha, \beta \rangle, \quad \langle \partial^{NCH} \check{\alpha}, \hat{\beta} \rangle = \langle -\partial^{H\mathcal{D}\mathcal{E}} \alpha, \beta \rangle.$$

If $\alpha$ is a bad Reeb orbit, then $\langle \partial^{NCH} \tilde{\alpha}, \hat{\beta} \rangle = 0$ for any Reeb orbit $\beta$; If $\beta$ is a bad Reeb orbit, then $\langle \partial^{NCH} \check{\alpha}, \hat{\beta} \rangle = 0$ for any Reeb orbit $\alpha$. $\langle \partial^{NCH} \check{\alpha}, \check{\beta} \rangle = 0$, except when $\alpha = \beta$ is bad, yielding a coefficient of -2.
\[ CC^S_*(Y, \lambda, J) = \bigoplus_{\alpha, k \geq 0} \mathbb{Z}\langle \tilde{\alpha} \otimes U^k, \hat{\alpha} \otimes U^k \rangle, \quad \partial^S = \partial^{NCH} \otimes 1 + \partial_1 \otimes U^{-1} \]

**Theorem (Hutchings-N. '19)**

When there exists a regular pair \((\lambda, J)\), meaning \(\partial^{EGH}\) is well-defined and \((\partial^{EGH})^2 = 0\), e.g. \(\dim(Y) = 3\), \(\lambda\) dynamically convex, \(J\) generic, then

\[ H_* \left( CC^S_*(Y, \lambda, J), \partial^S \right) \otimes \mathbb{Q} = CH^E_{EGH}(Y, \lambda, J). \]

1. Let \(C'_*\) be the submodule missing generators of the form \(\tilde{\beta} \otimes 1\) where \(\beta\) is good. Then \(C'_*\) is a subcomplex of \(CC^S_*(Y, \lambda)\).

2. \(H_* \left( C' \otimes \mathbb{Q}, \partial^S \otimes 1 \right) = 0.\)

3. \(CH^S_*(Y, \xi) \otimes \mathbb{Q} = H_* \left( \left( CC^S_*(Y, \lambda)/C'_* \right) \otimes \mathbb{Q}, \partial^S \otimes 1 \right).\)

A basis for this quotient complex is given by \(\tilde{\alpha} \otimes 1\), for \(\alpha\) good. The differential is induced by \(\tilde{\partial}\) & after tensoring w/\(\mathbb{Q}\) agrees with \(\partial^{EGH}.\)

**Corollary (Hutchings-N '19 (modulo obg))**

\(CH^{E_{GH}}_*(Y^3, \ker \lambda)\) does not depend on \(J\) or dynamically convex \(\lambda\)!
The period doubling bifurcation

Bifurcations of Reeb orbits occur as we deform $\lambda$. Let $\{\Phi_\tau\}_{\tau \in [0,1]} : D \rightarrow D$, be a partial return map

$$\Phi_\tau = \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \circ \varphi_{\epsilon \tau}^{X_\tau},$$

where $\varphi_{\epsilon \tau}^{X_\tau}$ is the $\epsilon$-flow of a $180^\circ$-rotation invariant $X_\tau$.

[Diagrams of elliptic and hyperbolic orbits]

- elliptic $E$ $\leadsto$ negative hyperbolic $h_1$
- rotation($E$) $\sim \frac{1}{2} - \epsilon$ $\leadsto$ new elliptic orbit $e_2$
- period($h_1$) $\sim$ period($E$), rotation($h_1$) $\sim$ rotation($E$)
- period($e_2$) $\sim 2 \cdot$ period($E$), rotation($e_2$) $\sim 2 \cdot$ rotation($E$)
Bifurcated $E$ has become

$$\Phi_\tau = \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \circ \varphi^{X_\tau}_\epsilon.$$ 

- $(r = 0)$ the critical point of $X_0$ is a fixed point of $\Phi_0$, corresponding to the **elliptic** orbit $E$,
- $(r = 1)$ the central critical point of $X_1$ is a fixed point of $\Phi_1$, corresponding to the **negative hyperbolic** orbit $h_1$.
- $(r = 1)$ the left and right critical points of $X_1$ are exchanged by $\Phi_1$, giving rise to a period 2 orbit, aka the **elliptic** orbit $e_2$. 

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Reflections on cylindrical contact homology
1. Fix an embedded Reeb orbit $\gamma$.

2. Locate other orbits winding $k$ times around tubular nhood $N_{\gamma}$ of $\gamma$.

3. Compute cylindrical contact homology in this tube.

Local $CH^E_{*} = H_{*}(\mathbb{Q} \langle \text{good orbits winding } k \text{ times around } N_{\gamma} \rangle, \partial|_{N_{\gamma}})$.

$(k = 2)$: $E^2$ is a generator before the bifurcation

$e_2$ is a generator after.

Even though $h_1^2$ winds twice around $N_{\gamma}$, it is a bad orbit,
and banished to the Sarlacc pits.

$$\text{Local } CH^E_{*}(\lambda_0, N_{\gamma}, 2) = \begin{cases} \mathbb{Q} & \text{if } * = 0 \text{ (generated by } E), \\ 0 & \text{otherwise.} \end{cases}$$

Local $CH^S_{*}$ sees more, rescuing the bad orbits!
Always two there are, the elliptic master and the hyperbolic apprentice.

Local $CH^{S_1}_* = H_*(\mathbb{Z} \langle \hat{\gamma}, \tilde{\gamma} \mid \gamma \text{ winds } k \text{ times around } N_\gamma \rangle \otimes \mathbb{Z}[U], \partial^{S_1}|_{N_\gamma})$

For $\lambda_0$, there is only one orbit in $N_\gamma$ which winds twice around: $E^2$

$$CH^{S_1}_*(\lambda_0, N_\gamma) = \begin{cases} 
\mathbb{Z} & \text{if } \ast = 0 \quad \text{(generated by } \tilde{E}), \\
\mathbb{Z}/2 & \text{if } \ast = 2k + 1 \quad \text{(generated by } u^k \tilde{E}), \\
0 & \text{otherwise}.
\end{cases}$$

For $\lambda_1$, there are two orbits in $N_\gamma$ which wind twice around: $e_2$ and $h^2$

$$CH^{S_1}_*(\lambda_1, N_\gamma) = \begin{cases} 
\mathbb{Z} & \text{if } \ast = 0 \quad \text{(generated by } \tilde{e}_2), \\
\mathbb{Z}/2 & \text{if } \ast = 2k + 1 \quad \text{(generated by } u^k \tilde{h}^2), \\
0 & \text{otherwise}.
\end{cases}$$

The 2-torsion before the bifurcation sees the bad Reeb orbit that can be created in the bifurcation!
Thanks!

Jo Nelson (Rice)  Reflections on cylindrical contact homology