Hodge theory aspects of homological mirror symmetry

Jingyu Zhao

Institute of Advanced Study

jzhao@ias.edu

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Hodge decomposition

Given a complex manifold, one can decompose the de Rham complex
\[ A_X^* := \Omega^*_d R(X) \otimes \mathbb{R} \mathbb{C} \] as \[ A^k_X \cong \bigoplus_{p+q=k} A^{p,q}(X), \] where \( \alpha \in A^{p,q}(X) \) is locally of the form
\[ \sum f_{i_1 \ldots i_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}. \]
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- For Kähler manifolds, Hodge theory gives the Hodge decomposition

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \cong \bigoplus_{p+q=k} H^p(X, \Omega^q_X),$$

where $\Omega^*_X$ is the sheaf of holomorphic differential forms. This decomposition depends on the complex structure.
Hodge-to-de Rham spectral sequence and $E_1$-degeneration

Let $X$ be a complex manifold, there is a double complex $A^\bullet \otimes \bar A^\bullet$, i.e. $\partial^2 = \bar \partial^2 = \partial \bar \partial + \bar \partial \partial = 0$. If $X$ is Kähler, the associated spectral sequence degenerates at $E_1$ and converges to de Rham cohomology $H^\bullet (X, \mathbb{C})$. In characteristic zero, $E_1$-degeneration follows from Hodge theory for Kähler manifolds. Deligne and Illusie gave another purely algebraic proof using reduction to finite characteristics. On $E_1$-page, $E_{p, q}^1 \hookrightarrow H^p (X, \Omega^q X)$ and $E_1$-degeneration implies the Hodge decomposition for $H^k (X, \mathbb{C})$. In fact, there is a (pure) Hodge structure of weight $k$ on $H^k (X, \mathbb{C})$. 

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The origin of mirror symmetry

A Calabi-Yau manifold is a complex manifold $X$ such that $K_X$ is trivial, i.e. there is a nonzero holomorphic volume form.

Mirror symmetry is first discovered for pairs of Calabi-Yau 3-folds, denoted as $X$ and $X_\_$. In 1990, Greene and Plesser constructed the mirror for the quintic 3-fold in $P_4$.

Candelas, de la Ossa, Green and Parkes predicted the genus zero Gromov-Witten invariants (symplectic) of $X$ using period integrals (complex) on the mirror $X_\_$ (Ref. Givental, Lian-Liu-Yau).
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Symmetry of Hodge diamonds

What does Hodge theory say about mirror pairs?

Let $h_{p,q}(X) := \dim \text{CH}^p(X, \mathcal{L}^q X)$.

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Mirror symmetry is manifested as a 90 degree rotation of Hodge diamonds.
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\[
\begin{array}{cccccccc}
1 & & & & & & & 1 \\
& 0 & 0 & & & & & \\
& & 0 & 1 & 0 & & & \\
& & & 101 & 101 & 1 & & \\
& & & & 1 & 1 & 1 & 1 \\
& & & & & 0 & 101 & 0 \\
& & & & & & 0 & 0 \\
& & & & & & & 1 \\
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- Given a symplectic manifold $X$ and a complex manifold $X^\vee$

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Categories

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Noncommutative Hodge-to-de Rham spectral sequence

Given an associative algebra $A$, on Hochschild chains $C^\bullet(A)$ one has two differentials, the Hochschild differential $b$ and Connes differential $B$, such that $bB + Bb = 0$.

$HH^\bullet(A) = H^\bullet(C^\bullet(A), b)$,
$HC^\bullet(A) = H^\bullet(C^\bullet(A), b + uB)$, $|u| = 2$.

If $A$ is the coordinate ring of a smooth affine variety $X$, then the Hochschild-Konstant-Rosenberg says $HH^\bullet(A) \hookrightarrow \implies \implies \Sigma^\bullet X$, and moreover $HKR : (HH^\bullet(A), B) \mapsto (\Sigma^\bullet X, d_{dR})$ is a map.

For associative algebra, or a differential graded (DG) category $A$ (such as $\text{Coh}(X)$), one can replace the Hodge-to-de Rham spectral sequence $H^p(X, \Sigma^q X) \mapsto$ by Hochschild-to-cyclic spectral sequence $HH^p(A) \hookrightarrow \implies HC^{p+q}(A)$.
Given an associative algebra $\mathcal{A}$, on Hochschild chains $C_\ast(\mathcal{A})$ one has two differentials, the Hochschild differential $b$ and Connes differential $B$, such that $bB + Bb = 0$. 
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Previous studies on Hodge theoretic aspects

Let $A$ be a smooth and proper DG category. (e.g. $\text{Coh}(X)$ of a projective variety, proper: morphisms space in $A$ has finite homological dimension.) Barannikov and Kontsevich-Katzarkov-Pantev have developed noncommutative Hodge theories for $A$. Kaledin in 2016 proved the degeneration of noncommutative Hodge-to-de Rham spectral sequence for smooth and proper DG categories. Ganatra, Perutz and Sheridan in 2015 used noncommutative Hodge theory to show that HMS for Calabi-Yau manifolds implies enumerative mirror symmetry for the quintic.
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Mirror symmetry for open manifolds

Prototype: The mirror pair $X = C^\times$ and $X_\_ = C^\times$. If one only allows compactly supported coherent sheaves in the category, then the only objects are skyscraper sheaves. It’s more natural to consider coherent sheaves with noncompact supports. E.g. Take the structure sheaf $O_{C^\times}$, the morphism space $\text{Ext}^\times(O_{C^\times}, O_{C^\times}) = C[z, z^{-1}]$ is infinite dimensional.

In order for HMS to hold, one needs a version of Fukaya category which is possibly nonproper. This is the wrapped Fukaya category $W(X)$ (Abouzaid-Seidel).

For open manifolds $U = X \setminus D$ where $X$ is a compact Kähler manifold and $D$ is a normal crossing divisor, Deligne in 1971 constructed a mixed Hodge structure on $H^\bullet(U, \mathbb{C}) = H^\bullet(X, \mathcal{I}^\bullet_X(\log D))$. 

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Q1: When does the Hodge-to-de Rham (Hochschild to cyclic homology) spectral sequence degenerate for $\mathcal{W}(X)$?

Q2: If it does not degenerate at $E_1$-page, does it degenerate at $E_2$-page or so on?

Q3: Given a (nondegenerate) Liouville manifold, by Ganatra $\text{HH}^\bullet(W(M)) \hookrightarrow = \text{SH}^\bullet + n(M)$ and $\text{HC}^\bullet(W(M)) \hookrightarrow = \text{SH}^\bullet + nS_1(M)$. When the spectral sequence degenerate at $E_1$, it induces a "Hodge" filtration. It is a symplectic invariant, does it respect symplectomorphisms?
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Q1: When does the Hodge-to-de Rham (Hochschild to cyclic homology) spectral sequence degenerate for $\mathcal{W}(X)$? By Kaledin, degenerate if $\mathcal{W}(M)$ is proper.

Q2: If it does not degenerate at $E_1$-page, does it degenerate at $E_2$-page or so on?

Q3: Given a (nondegenerate) Liouville manifold, by Ganatra $HH_*(\mathcal{W}(M)) \cong SH^{*+n}(M)$ and $HC_*(\mathcal{W}(M)) \cong SH^{*+n}_{S^1}(M)$. When the spectral sequence degenerates at $E_1$, it induces a "Hodge" filtration. It is a symplectic invariant, does it respect symplectomorphisms?
Thank you!