Spectra of non-normal random matrices and noise stability

Ofer Zeitouni
Weizmann Institute
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Consider the nilpotent $N$-by-$N$ matrix

$$T_N = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & 1 \\
0 & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}$$

Eigenvalues $\lambda_i = 0$, empirical measure $n^{-1} \sum \delta_{\lambda_i} = \delta_0$. Let $G_N$ be a matrix with i.i.d. standard Gaussians. For $\gamma > 1/2$, $\|N^{-\gamma} G_N\| \to 0$, almost surely.
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Theorem (Guionnet-Wood-Z. ’11)

Set \( A_N = T_N + N^{-\gamma} G_N \), eigenvalues \( \eta_i \), empirical measure \( L^A_N = n^{-1} \sum \delta_{\eta_i} \). \( \gamma > 1/2 \). Then \( L^A_N \) converges weakly to the uniform measure on the unit circle in the complex plane.

Thus, \( L^T_N = \delta_0 \) but for a vanishing perturbation, \( L^A_N \) has different limit. (Generalization to i.i.d. \( G_N \): Wood ’13.)
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$A_N$ - $N \times N$ matrix, uniformly bounded in operator norm.

Definitions: $A_N$ converges in $*$-moments toward an element $a$ in a $\mathcal{W}^*$
probability space $(\mathcal{A}, \| \cdot \|, *, \phi)$ (faithful trace $\phi$) if for any non-commutative polynomial $P$,$\frac{1}{N} \text{tr} P(A_N, A_N^*) \xrightarrow{N \to \infty} \phi(P(a, a^*))$.

Fuglede–Kadison Determinant $\det(a) = \exp(\phi(\log |a|))$.

Brown measure $\nu_a$ of $a \in \mathcal{A}$:

$$\log \det(z - a) = \int \log |z - z'| d\nu_a(z'), \quad z \in \mathbb{C}. $$

Given by

$$\nu_a(dz) = \frac{1}{2\pi} \Delta_z \log(\det(a - z)).$$

In particular $\log \det(z - a) = \int \log x \ d\nu^Z_a(x) \quad z \in \mathbb{C}$, where $\nu^Z_a$ denotes the spectral measure of the operator $|z - a|$. 
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Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

**Theorem (Śniady '02)**

$$\lim_{t \to 0} \lim_{N \to \infty} L_N^{A_N(t)} = \nu a.$$  

In particular, some sequence of noise regularizes empirical measure to the Brown measure.

Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of $\Sigma$, $\Sigma_A$, for $f$ coordinate-wise increasing,

$$N^{-1} \text{tr}(f(\Sigma_A(t))) \geq N^{-1} \text{tr}(f(\Sigma_0(t))).$$

This gives required control of the determinant; Second part of theorem follows by diagonalization argument.

How can we take $t = t_N \to 0$?
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\( \mathbf{a} \in \mathcal{A} \) is regular if for \( f \) smooth, compactly supported,

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\lim_{\epsilon \to 0} \int_{\mathbb{C}} \Delta \psi(z) \left( \int_{0}^{\epsilon} \log x \, d\nu^{z}_{a}(x) \right) \, dz = 0
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Theorem (Guionnet-Wood-Z. ’11)

Assume: \( A_{N} \to^{*} a \), regular. \( L_{N}^{A} \to \nu_{a} \) weakly. \( \gamma > 1/2 \). Then,

\( L_{N}^{A_{N} + N^{-\gamma} G_{N}} \to \nu_{a} \) weakly, in probability.

The proof uses the regularity (of the limit) to truncate the singularity of the log... and depends crucially on convergence to \( \nu_{a} \). But it is not useful in maximally nilpotent example, since \( L_{N}^{A} = \delta_{0} \not\to \nu_{a} = \delta_{S^{1}} \).
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Assume: $A_N \to^* a$, regular, $\|E_N\| \to 0$ polynomially. $L_N^{A_N + E_N} \to \nu_a$ weakly. Then $L_N^{A_N + (N^{-\gamma} G_N)} \to \nu_a$ weakly, in probability.

So it is enough to find a perturbation with correct limiting behavior! Nilpotent example uses $a$- unitary element (which is regular), $E_N$ is $(N, 1)$ element.
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Noise Stability-Maximal Nilpotent

\[ L_N^{A_N + E_N} \rightarrow S^1 \]

**eigenvalues**

\[ E_N^{1/N} \]
Noise Stability - Nilpotent matrices

Maybe this always works?

\[ T_{b,N} = \begin{bmatrix} T_b & & & \\ & T_b & & \\ & & \ddots & \\ & & & T_b \end{bmatrix} \]

where \( T_b \) is maximally nilpotent of dimension \( b \).

Theorem (Guionnet-Wood-Z '11)

If \( b = a \log N \) and \( \gamma \) is large enough, then the spectral radius of \( T_{b,N} + N^{-\gamma} G_N \) is uniformly strictly smaller than 1. In particular,

\[ L_N^{T_{a \log N, N} + N^{-\gamma} G_N} \not\xrightarrow{} \delta_{S^1} \]

even though \( T_{a \log N, N} \) converges in \( \ast \) moments to random unitary!

What is going on?
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where \( T_b \) is maximally nilpotent of dimension \( b \).

Theorem (Guionnet-Wood-Z ’11)

If \( b = a \log N \) and \( \gamma \) is large enough, then the spectral radius of 
\( T_{b,N} + N^{-\gamma} G_N \) is uniformly strictly smaller than \( 1 \). In particular,

\[
L_N^{T_{a \log N, N+N^{-\gamma} G_N}} \not\to \delta_{S^1}
\]

even though \( T_{a \log N, N} \) converges in \(*\) moments to random unitary!

What is going on?
Noise Stability-Nilpotent matrices

Maybe this always works?

\[ T_{b,N} = \begin{bmatrix} T_b & & \\ & T_b & \\ & & \ddots \ & T_b \end{bmatrix} \]

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Ofer Zeitouni
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**What is going on?**
Simulations inconclusive!

\[
\text{In block: } \left( N^{-\sigma} \right)^{\alpha \log n} \sim e^{-\alpha a}
\]
Simulations inconclusive!

In block: \( (N^{-\frac{1}{2}})^{\log N} \approx e^{-\frac{1}{\sqrt{N}}} \)

\( Q \) \( \delta \) \( e^{-\frac{1}{\sqrt{N}}} \)
Framework: $B^i = B^i(N)$ - Jordan blocks, dimension $a_i(N) \log N$, eigenvalue $c_i(N)$.

$$A_N = \begin{bmatrix} B^1 & \ & \ \\ & B^2 & \ \\ & & \ldots \ \\ & & & B^\ell(N) \end{bmatrix}.$$
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Simulations...
Simulations inconclusive!

\[ \Re z \gamma = 1.0 \]

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$A_N$ block matrix, each block of size $a_i \log N$. $c_i$ on diagonal.

$B_N = A_N + N^{-\gamma} G_N$.

Define $r_i(N) = e^{(-\gamma+1/2)/a_i} \leq 1$. Set $\mu_N = \frac{1}{N} \sum_{i=1}^{\ell(N)} a_i \log N \nu_{c_i, r_i}$ where $\nu_{c, r}$ uniform on circle of radius $r$ centered on $c$.

Theorem (Feldheim, Paquette, Z. ’14)

For $\gamma > 1$ and $\ell(N) = o(N/\log \log(N))$,

$$d(L_{B_N}^{B_N}, \mu_N) \to_{N \to \infty} 0$$

Analogous result for $\gamma \in (1/2, \]$ if collection of circles “does not spread too much” (e.g., olympics rings example OK).
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Analogous result for $\gamma \in (1/2, \] if collection of circles “does not spread too much” (e.g., olympics rings example OK).
Again, logarithmic potential plays a crucial role in the proof.
By general results, enough to show that for Lebesgue a.e. \( z \),

\[
|U_{LB}^N(z) - U_{\mu_N}(z)| \to 0,
\]

in probability, where \( U_\nu(z) = \int \log |z - x| \nu(dx) \).
For \( L_N^B \), \( U_{LB}^N(z) = \frac{1}{2N} \log \det(z - B_N)(z - B_N)^* \).
In estimating it, an important role is played by lower bounding the determinant of \( A + G_n \) independently of \( A \), for appropriate \( n \leq N \).
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For $L^B_N$, $U_{LB}(z) = \frac{1}{2N} \log \det(z - B_N)(z - B_N)^\ast$.

In estimating it, an important role is played by lower bounding the determinant of $A + G_n$ independently of $A$, for appropriate $n \leq N$. 
Sketch of UB: all blocks equal, $c=0$

Consider separately $|z_1|$ small & $|z_1|$ large.

$$\det \begin{pmatrix} Z_{-1} & 0 \\ 0 & Z \end{pmatrix} = \frac{1}{Z^N} \det \begin{pmatrix} Z_{-1} & \xi \\ \xi^T & 1 \end{pmatrix}$$

Expand in minors: main minors $= 1$, $z_{-1}$ minors:

$$(Z_{-1})^N \cdot (N^{-\delta})^2 \cdot \det (G)_{2 \times 2}$$

$$\approx (Z_{-1})^N \cdot N^{-\delta} \cdot N^{1/2} (1+o(1))$$

For $|z_1|$ small, competition between main diag & off diag - better win.

For $|z_1|$ large - 1 wins. Cutoff at scales.
As in UB, perform row and column permutations and then write

\[ B_N - zI = \begin{bmatrix} T + G_1 & * \\ * & G_2 \end{bmatrix} , \]

We need to fight cancelations between possible contributions to the determinant. Using Schur complement,

\[ \det(B_N - zI) = \det(T + G_1) \det(G_2 - C) \]

For Gaussian matrices \( G_2 \), easy to bound second determinant from below, independently of \( C \), by height \( \times \) area formula. For non-Gaussian noise, no general estimates for minimum singular values if \( C \) is arbitrary (i.e. no prior assumption on norm of \( C \! \)).
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