MATRIX MODELS,
LAPLACIAN GROWTH
AND
HURWITZ NUMBERS

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have a common integrable structure which is

2D Toda lattice hierarchy in zero dispersion limit
Normal random matrices

$$\left[ M, M^\dagger \right] = 0$$

Partition function

$$Z_N = C_N \int_{N \times N} DM \exp \left( \frac{1}{\hbar} \text{tr} W(M, M^\dagger) \right)$$

Potential

$$W(M, M^\dagger) = -U(M, M^\dagger) + \sum_{k \geq 1} \left( t_k M^k + \bar{t}_k (M^\dagger)^k \right)$$

Example:

$$U = MM^\dagger$$
Passing to eigenvalues

\[ Z_N(\{t_j\}, \{\bar{t}_j\}) = \frac{1}{N!} \int_{C} \cdots \int_{C} \prod_{m < n} |z_m - z_n|^2 \prod_{j=1}^{N} e^{-\frac{1}{\hbar} U(z, \bar{z}) + \frac{1}{\hbar} \sum_{k \geq 1} (t_k z_j^k + \bar{t}_k \bar{z}_j^k)} d^2 z_j \]

\[ = \tau_N(\{t_j\}, \{\bar{t}_j\}) \]

**Integrability:** \( \tau_N(\{t_j\}, \{\bar{t}_j\}) \) is tau-function of the 2D Toda

**Dispersionless limit = large \( N \) limit**

\[ N \to \infty, \quad \hbar \to 0 \quad \text{with} \quad Nh = t_0 = t \text{ fixed} \]

\[ \tau_N(\{t_j\}, \{\bar{t}_j\}) = \exp \left( \frac{1}{\hbar^2} (F(t_0, \{t_j\}, \{\bar{t}_j\}) + O(\hbar)) \right) \]
The leading large $N$ approximation: the integral is determined by maximum of the integrand

Microscopic density of eigenvalues

$$\rho(z) = \hbar \sum_{j=1}^{N} \delta^{(2)}(z - z_j)$$

Support of eigenvalues (assuming it is simply-connected):

A domain $D$ such that

$$\lim_{N \to \infty} \langle \rho(z) \rangle > 0 \quad \text{if} \quad z \in D$$

and 0 otherwise

$$\lim_{N \to \infty} \langle \rho(z) \rangle = \frac{1}{\pi} \partial \overline{\partial} U(z, \overline{z}), \quad z \in D$$
The shape of $D$ is a solution of the inverse potential problem

Example: $W(z) = -|z|^2 + 2\Re \sum_{k>0} t_k z^k$

$$\langle \rho(z) \rangle = \frac{1}{\pi}, \quad z \in D \quad \text{and} \quad 0 \quad \text{otherwise}$$

$$t_k = -\frac{1}{\pi k} \int \int_{\text{ext. of } D} z^{-k}d^2z$$

$$t_0 = t = Nh = \frac{\text{area } (D)}{\pi}$$
We restart with a set of data in the complex plane.

A function $U(z, \bar{z})$ in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that

$$\sigma(z, \bar{z}) := \partial \bar{\partial} U(z, \bar{z}) > 0$$

(background charge density, conformal metric, ...)

This function parametrizes solutions to the 2D Toda hierarchy.

Example 1

$$U(z, \bar{z}) = z \bar{z}, \quad \sigma(z, \bar{z}) = 1$$

Example 2

$$U(z, \bar{z}) = \frac{1}{2\beta} \left[ \log \frac{z \bar{z}}{Q} \right]^2 \quad \sigma(z, \bar{z}) = \frac{1}{\beta z \bar{z}}$$
A simply-connected compact domain $D$ in the complex plane with smooth boundary $\gamma$

Moments of the exterior:

$$t_k = \frac{1}{2\pi i k} \oint_\gamma z^{-k} \partial U(z, \bar{z}) \, dz = -\frac{1}{\pi k} \iint_{D^c} z^{-k} \sigma(z, \bar{z}) \, d^2 z, \quad k \geq 1$$

$$t_0 = \frac{1}{2\pi i} \oint_\gamma \partial U(z, \bar{z}) \, dz = \frac{1}{\pi} \iint_D \sigma(z, \bar{z}) \, d^2 z$$
Complimentary set of moments (moments of the interior):

\[ v_k = \frac{1}{2\pi i} \oint_{\gamma} z^k \partial U(z, \bar{z}) \, dz = \frac{1}{\pi} \iint_{D^c} z^k \sigma(z, \bar{z}) \, d^2 z, \quad k \geq 1 \]

Logarithmic moment:

\[ v_0 = \frac{1}{\pi} \iint_{D} \log |z|^2 \sigma(z, \bar{z}) \, d^2 z. \]
Potential created by the charge in D

\[ \Phi(z, \bar{z}) = -\frac{2}{\pi} \int_D d^2 z' \sigma(z', \bar{z}') \log |z - z'| \]

Expansion inside

\[ \Phi^+(z, \bar{z}) = -U(z, \bar{z}) + v_0 + 2\Re \sum_{k>0} t_k z^k \]

Expansion outside

\[ \Phi^-(z, \bar{z}) = -2t_0 \log |z| + 2\Re \sum_{k>0} \frac{v_k}{k} z^{-k} \]
Theorem

The real parameters $t_0, \text{Re} t_k, \text{Im} t_k, k \geq 1$

are local coordinates in the space of simply connected domains with smooth boundary

This means:

1. Any one-parameter deformation $D(t)$ of $D = D(0)$ with some real parameter $t$ such that $\partial_t t_k = 0, k \geq 0$ is trivial (local uniqueness of domain with given moments)

2. These parameters are independent

In particular, moments $v_k$ are functions of $t_k$.
Green's function of the Dirichlet boundary value problem in the exterior of $D$

\[ G(z, \xi) = \frac{1}{2\pi} \log |z - \xi| + g(z, \xi) \]

- $G(z, \xi) = G(\xi, z)$ and $G(z, \xi') = 0$ for any $z \in D^c$ and $\xi' \in \gamma$.
- The function $g(z, \xi)$ is harmonic in $z$ for any $\xi \in D^c$.

It gives universal solution to the Dirichlet boundary value problem

\[ u(z) = -\int_{\gamma} u_0(\xi) \partial_{n_\xi} G(z, \xi) \, d\xi \]

(the Poisson formula)
Conformal map

We normalize $w(z)$ by the conditions

\[ G(z, \xi) = \frac{1}{2\pi} \log \left| \frac{w(z) - w(\xi)}{w(z)w(\xi) - 1} \right| \]

**We normalize $w(z)$ by the conditions**

\[ w(\infty) = \infty \text{ and } w'(\infty) \text{ is real positive} \]

\[ w(z) = pz + \sum_{j \geq 0} p_j z^{-j}, \text{ where } p > 0 \]
Infinitesimal deformations can be described by normal displacement of the boundary

\[ \delta n (\xi) \]

\[ \xi \]

Special deformations

\[ \delta_a n(z) = -\frac{\varepsilon \pi}{\sigma(z, \bar{z})} \partial_{n_z} G(a, z), \quad z \in \gamma, \quad \varepsilon \to 0. \]
Consider the function

\[ H_k(\xi) = -i \int_\infty z^k \partial_z G(z, \xi) \, dz \]

and the deformations

\[ \delta n(\xi) = \varepsilon \Re (\partial_{n\xi} H_k(\xi)) \quad \text{and} \quad \delta n(\xi) = \varepsilon \Im (\partial_{n\xi} H_k(\xi)) \]

They change \( x_k = \Re t_k \) and \( y_k = \Im t_k \) only

\[ \delta_{\infty} n(\xi) = -\frac{\varepsilon \pi}{\sigma(\xi, \bar{\xi})} \partial_{n\xi} G(\infty, \xi) \quad \text{changes } t_0 \text{ only.} \]
Introduce differential operators

\[ D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k, \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k \]

where \( \partial_k = \partial/\partial t_k, \bar{\partial}_k = \partial/\partial \bar{t}_k \)

and the operator \( \nabla(z) = \partial_0 + D(z) + \bar{D}(\bar{z}) \)

Lemma

Let \( X \) be any functional on the set of domains \( D \) regarded as a function of \( t_0, \{t_k\}, \{\bar{t}_k\} \), then for any \( z \) in the exterior of \( D \) we have

\[ \delta_z X = \varepsilon \nabla(z) X \]
The dispersionless tau-function

\[ F = -\frac{1}{\pi^2} \iiint_D \iiint_D \sigma(z, \bar{z}) \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) \, d^2z \, d^2\zeta \]

**Theorem**

\[ \nabla(z) F = -\frac{2}{\pi} \iint_D \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) \, d^2\zeta \]

**Corollary**

\[ v_0 = \partial_0 F, \quad v_k = \partial_k F, \quad \bar{v}_k = \bar{\partial}_k F, \quad k \geq 1 \]
Theorem

\[ G(z, \zeta) = \frac{1}{2\pi} \log |z^{-1} - \zeta^{-1}| + \frac{1}{4\pi} \nabla(z) \nabla(\zeta) F. \]

Corollary

The conformal map \( w(z) \) is given by

\[ w(z) = z \exp \left( \left( -\frac{1}{2} \partial_0^2 - \partial_0 D(z) \right) F \right) \]
Theorem

The function $F$ satisfies

\[
(z - \xi) e^{D(z)D(\xi)F} = ze^{-\partial_0 D(z)F} - \xi e^{-\partial_0 D(\xi)F}
\]
\[
(\bar{z} - \bar{\xi}) e^{\bar{D}(\bar{z})\bar{D}(\bar{\xi})F} = \bar{z}e^{-\partial_0 \bar{D}(z)F} - \bar{\xi} e^{-\partial_0 \bar{D}(\bar{\xi})F}
\]
\[
1 - e^{-D(z)\bar{D}(\bar{\xi})F} = \frac{1}{z\xi} e^{\partial_0 (\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}
\]

(These are equations of the dispersionless 2D Toda hierarchy in the Hirota form)
Important comment:

Although the definitions of the moments and the function $F$ depend on the background density, the formulas for the Green's function and the conformal map do not.

This means that the conformal maps can be described by any non-degenerate solution of the Toda hierarchy.
Physical applications to Hele-Shaw flows (Laplacian growth)

\[ U(z, \bar{z}) = z\bar{z}, \quad \sigma(z, \bar{z}) = 1 \]

Then the vector field in the space of domains corresponding to the special deformation with the Green function \( G(z, a) \) is the **Hele-Shaw flow** (with zero surface tension) with a sink at the point \( a \)

\[ V_n(z) \propto \partial_{n_2} G(z, a) \]

In particular, \( a \to \infty, \quad V_n \propto \partial_n \log |w(z)| \)
The Hele-Shaw cell

The Darcy law: \( \vec{V} = -\vec{\nabla} \Phi \) \( \Delta \Phi(Z) = 0 \)

At the interface: \( V_n(Z) = -\partial_n \Phi(Z) \)
Radial Hele-Shaw cell, schematic view
Experimental patterns

Large flux, small surface tension (after Swinney)

Small flux, large surface tension
Laplacian growth and inverse potential problem

(S. Richardson, 1972)

Exterior harmonic moments are conserved.

\[ t_k = \frac{1}{2\pi ik} \oint \gamma z^{-k} \bar{z} dz \]

\[ t_0 = \text{Area } (D)/\pi = \text{time} \]

The LG process is changing the area keeping the harmonic moments constant.
Integrability of the radial Laplacian Growth problem

(M. Mineev-Weinstein, P. Wiegmann, A.Z., 1999)

Dispersionless tau-function

\[ F = F(t_0, \{ t_k \}, \{ \bar{t}_k \}) \]

Conformal map from the domain to the exterior of the unit disk

\[ w(z) = z \exp \left( -\frac{1}{2} \partial_{t_0}^2 F_0 - \partial_{t_0} D(z) F \right) \]

The Green's function

\[ G(z, \zeta) = \log |z^{-1} - \zeta^{-1}| + \frac{1}{2} \nabla(z, \bar{z}) \nabla(\zeta, \bar{\zeta}) F, \]

\[ \nabla(z, \bar{z}) := \partial_{t_0} + \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k} + \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k} = \partial_{t_0} + D(z) + \bar{D}(\bar{z}) \]
The Hele-Shaw problem in a channel

- is integrable (can be embedded in the same Toda hierarchy)

- is related to algebraic geometry of ramified coverings
\[ \begin{aligned}
\Delta \Phi(Z) &= 0 \quad \text{in} \quad D_+ \\
\Phi(Z + 2\pi i R) &= \Phi(Z) \\
\Phi(Z) &= 0, \quad Z \in \Gamma \\
\Phi(Z) &= -\frac{1}{2} \Re Z + \ldots \quad \text{as} \quad \Re Z \to +\infty
\end{aligned} \]
Conformal map:

\[ W(Z) = \frac{Z}{R} + \sum_{k \geq 0} c_k e^{-kZ/R} \]

Solution:

\[ \Phi(Z) = -\frac{R}{2} \Re e W(Z) \]

\[ V_n(Z) = \frac{R}{2} |W'(Z)| \]
Physical plane and auxiliary physical plane

\[ Z = R \log \left( \frac{z}{r_0} \right) \]

\[ W = \log w \]
We can express the normal velocity in the physical plane through the normal velocity in the auxiliary physical plane:

\[ V_n(Z) = \left| \frac{dZ}{dz} \right| V_n(z) = \frac{R}{|z|} V_n(z) \]

If \( U(z, \bar{z}) = \frac{R}{2} \left[ \log \frac{z \bar{z}}{r_0^2} \right]^2 \), then

\[ V_n(z) (z) = \frac{|z|^2}{2R} |w'(z)| , \quad z \in \gamma \]

\[ V_n(Z) (Z) = \frac{R}{|Z|} V_n(z) (z) = \frac{R}{2} |W'(Z)| \]
The dispersionless tau-function

\[ F_0 = -\frac{R^2}{\pi^2} \iint_{D \setminus B(r_0)} \iint_{D \setminus B(r_0)} \log |z^{-1} - \zeta^{-1}| \frac{d^2 z d^2 \zeta}{|z \zeta|^2} \]

\[ = -\frac{1}{\pi^2 R^2} \iint_{D^{(0)}} \iint_{D^{(0)}} \log |e^{-Z/R} - e^{-Z'/R}| d^2 Z d^2 Z' - t_0^2 \log r_0 \]

\[ \quad \frac{2F_0 = R \partial_R F_0 + t_0 \partial_{t_0} F_0 + \sum_{k>1} (t_k \partial_{t_k} F_0 + \bar{t}_k \partial_{\bar{t}_k} F_0)}{\beta = 1/R,} \]

\[ \partial_\beta F_0 = \frac{t_0^3}{6} + t_0 \sum_{k>1} k t_k \partial_{t_k} F_0 \]

\[ + \frac{1}{2} \sum_{k, l \geq 1} (k l t_k t_l \partial_{t_{k+l}} F_0 + (k + l) t_{k+l} \partial_{t_k} F_0 \partial_{t_l} F_0) \]

(the cut-and-join operator)
Example

\[ t_0 = t, \ t_1 \neq 0, \ t_k = 0 \at k \geq 2. \]

\[ Z(W) = RW + u_0 + u_1 e^{-W} \]

Trochoid

Cycloid
Hurwitz numbers

degree $d$ covering

$f : \Sigma \longrightarrow \mathbb{C}P^1$

$l$ simple ramification points

\[ \mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}) \]

\[ d = \sum_{i=1}^{\ell(\mu)} \mu_i := |\mu| \]

\[ |\mu| = |\bar{\mu}| = d \]

double Hurwitz numbers $H_{d,l}(\mu, \bar{\mu})$
Generating function of the double Hurwitz numbers for connected coverings

\[ t = \{t_1, t_2, \ldots\}, \quad \overline{t} = \{\overline{t}_1, \overline{t}_2, \ldots\} \]

\[
F^{(H)}(\beta, Q, t, \overline{t}) = \sum_{l \geq 0} \frac{\beta^l}{l!} \sum_{d \geq 1} Q^d \sum_{|\mu| = |\overline{\mu}| = d} H_{d,l}(\mu, \overline{\mu}) \prod_{i=1}^{\ell(\mu)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\overline{\mu})} \overline{\mu}_i \overline{t}_{\overline{\mu}_i}
\]

\[
\tau_n(t, \overline{t}) = e^{\frac{1}{12} \beta n(n+1)(2n+1)} Q^{\frac{1}{2} n(n+1)} \exp\left( F^{(H)}(\beta, e^{\beta(n+\frac{1}{2})} Q, t, \overline{t}) \right)
\]

is the tau-function of the 2D Toda lattice hierarchy

(A. Okounkov, 2000)
Genus expansion

\[ t_k \rightarrow t_k / \hbar, \ \beta \rightarrow \hbar \beta \]

\[ F^{(H)}(\hbar; \beta, Q, t, \bar{t}) := \hbar^2 F^{(H)}(\hbar \beta, Q, t / \hbar, \bar{t} / \hbar) \]

\[ F^{(H)}(\hbar; \beta, Q, t, \bar{t}) = \sum_{g \geq 0} \hbar^{2g} F_g^{(H)}(\beta, Q, t, \bar{t}) \]

Riemann-Hurwitz formula

\[ 2g - 2 = l - \ell(\mu) - \ell(\bar{\mu}) \]
The generating function of double Hurwitz numbers for connected genus 0 coverings

\[ F_0^{(H)} = \sum_{d \geq 1} \sum_{|\mu|=|\bar{\mu}|=d} \frac{Q^d}{\beta^2 (\ell(\mu) + \ell(\bar{\mu}) - 2)!} \frac{H_{d,\ell(\mu)+\ell(\bar{\mu})-2}(\mu, \bar{\mu})}{\prod_{i=1}^{\ell(\mu)} (\beta \mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar{\mu})} (\beta \bar{\mu}_i \bar{t}_{\bar{\mu}_i})} \]

Relation to the dispersionless tau-function for LG on a cylinder

\[ \beta = 1/R, \quad Q = r_0^2. \]

\[ F_0 = \frac{\beta t_0^3}{6} + t_0^2 \log r_0 + F_0^{(H)}(\beta, r_0^2 e^{\beta t_0}, t, \bar{t}) \]
Conclusion:

Conformal maps of plane domains and connected genus 0 ramified coverings of the sphere are governed by the same "master function" which is a special solution to the dispersionless Toda hierarchy.

Some references
