• My name is **Shay Moran**

• Before coming to IAS, I spent a year in California
  • at **UCSD** hosted by **Shachar Lovett**
  • at the **Simons Institute in Berkeley**

• I did my Ph.D. at the **Technion**, supervised by **Amir Yehudayoff** and **Amir Shpilka**.

• Research interests:  math ∩ computer-science
  • machine learning and its links with other fields

• Today I will present a problem in **abstract convexity**
Weak epsilon nets versus the Radon number

based on joint work with Amir Yehudayoff (Technion)
Weak epsilon nets for convex sets


For every distribution $\mu$ on $\mathbb{R}^d$ and $\epsilon > 0$ there is a set $N$ s.t:

1. $N \cap C \neq \emptyset$ for every convex $C$ with $\mu(C) \geq \epsilon$.
2. $|N| \lesssim (1 / \epsilon)^d$.

- $N$ is called a weak $\epsilon$-net w.r.t $\mu$
- bound does not depend on $\mu$!
- used in Alon & Kleitman’s proof of the $(p,q)$-conjesture by Hadwiger & Debrunner
- some conjecture the right bound is some $\tilde{O}(d / \epsilon)$
An example in the plane

Given: $\mu$ is the uniform distribution on $m$ points, $\epsilon > 0$

Objective: find a small $\epsilon$-net w.r.t $\mu$.

(hit every convex set with at least 6 blue points)
What properties of convex sets enable weak epsilon nets?

- Proofs use a lot of geometry
- Goal: identify a simple/basic property of convex sets that captures existence of weak epsilon nets
- notion of dimension?
- It is convenient to consider the framework of *abstract convexity* [Levi ‘51, Danzer-Grünbaum-Klee ’63, Kay-Womble ‘71]
Abstract convexity spaces

Definition. A convexity space is a pair \((X,F)\) where \(F\) is a family of subsets of \(X\) that is closed under intersections.

Some convexity spaces:

- closed/compact sets
- subgroups/subfields/subrings
- euclidean convex sets, grid convex sets
- geodesic convex sets
- ...

[“Math is the art of giving the same name to different things” – Henri Poincaré]

Definition. \((X,F)\) has weak \(\epsilon\)-nets if there is \(n=n(\epsilon)\) s.t: for every distribution \(\mu\) there are \(n\) points that hit all sets in \(F\) with measure at least \(\epsilon\).
The Radon number of abstract convexity spaces

**Theorem.** [Radon ’21]
Any $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect.

**Definition.** The *Radon number* of $(X,F)$ is the smallest integer $r$ s.t any set of $r$ points can be partitioned into two sets whose convex hulls intersect.
Conjecture: *Radon number* captures weak epsilon nets

**Conjecture.**
Let \((X,F)\) be a convexity space. The following are equivalent

1. \((X,F)\) has a finite radon number.
2. \((X,F)\) has a weak \(\varepsilon\)-nets of for every \(\varepsilon > 0\).

\[ 2 \rightarrow 1 \] is known \([M-Yehudayoff \ '17]\).

- proof by a simple reduction to the *chromatic number of Kneser graphs*
  - analyzing the chromatic number of Kneser graphs \([Lovász \ '78]\) is one of the first applications of the *topological method* in combinatorics.
Theorem: *Radon number* captures weak epsilon nets for separable spaces

**Theorem.** [M-Yehudayoff ‘17]
Let \((X,F)\) be a *separable* convexity space. The following are equivalent
1. \((X,F)\) has a finite radon number.
2. \((X,F)\) has a weak \(\epsilon\)-nets of finite size for every \(\epsilon > 0\).

**Definition.** Let \((X,F)\) be a convexity space. A *hyperplane* is a partition of \(X\) into two convex sets (members of \(F\)).

**Definition.** \((X,F)\) is *separable* if for every \(C \in F\) and \(x \notin C\) there is a *hyperplane* separating \(x\) and \(C\). [abstraction of Hahn-Banach]
Remains open: does a finite Radon number imply weak epsilon nets in non-separable spaces?

**Example.** (non-separable convexity space)

Let $G$ be a group with identity $e$.

Consider the convexity space $(G,F)$, where

$$F = \{ H \setminus \{e\} : H \leq G \}.$$  

Radon number $r$ means:
any $r$ elements can be partitioned to two sets $A,B$ whose generated groups $<A>$, $<B>$ share an element $x \neq e$.  

Thank you!