- My name is **Shay Moran**

- Before coming to IAS, I spent a year in California
  - at **UCSD** hosted by **Shachar Lovett**
  - at the **Simons Institute in Berkeley**

- I did my Ph.D. at the **Technion**, supervised by **Amir Yehudayoff** and **Amir Shpilka**.

- Research interests: math $\cap$ computer-science
  - machine learning and its links with other fields

- Today I will present a problem in ***abstract convexity***

Weak epsilon nets versus the Radon number

based on joint work with

Amir Yehudayoff (Technion)

Weak epsilon nets for convex sets

Theorem. *[Bárány-Füredi-Lovász '90, Alon-Bárány-Füredi-Kleitman '92, Chazelle-Edelsbrunner-Grigni-Guibas-Sharir-Welzl '93]*

For every distribution μ on $\mathbb{R}^d$ and $\epsilon > 0$ there is a set N s.t:

1. $N \cap C \neq \emptyset$ for every <u>convex</u> C with $\mu(C) \geq \epsilon$.

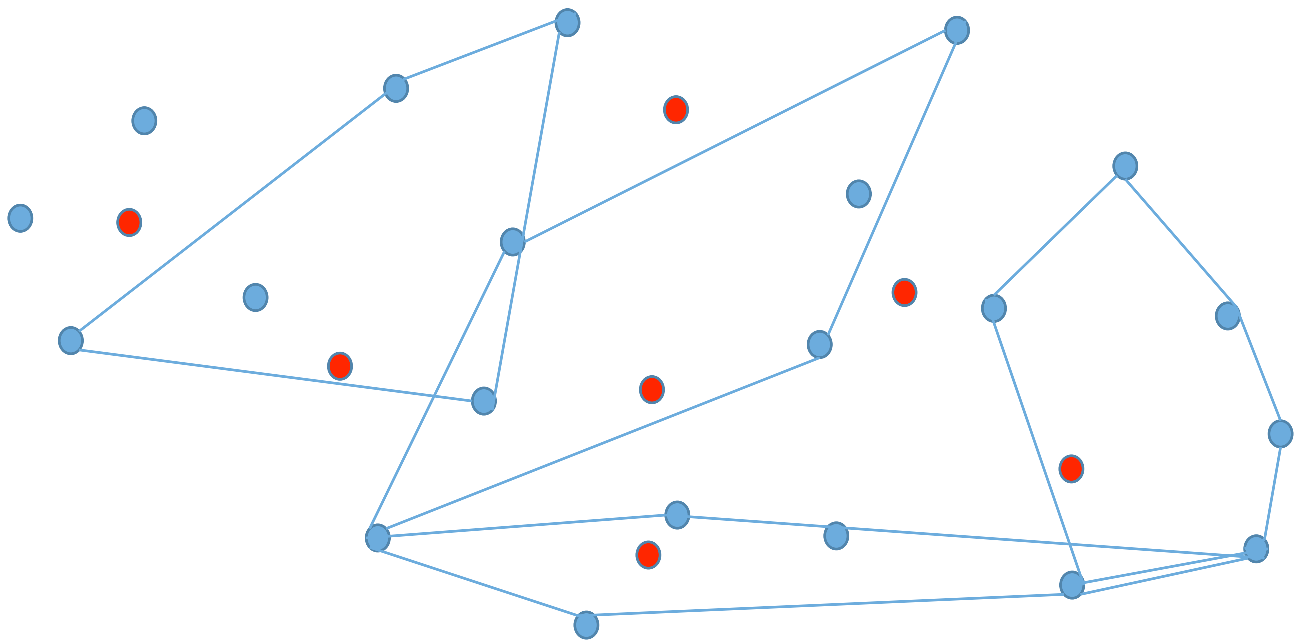
2. $|N| \lesssim (1/\epsilon)^d$.

- N is called a weak ϵ-net w.r.t μ
- bound ***does not depend*** on μ!
- used in Alon & Kleitman's proof of the ***(p,q)*-conjeture by Hadwiger & Debrunner**
- some conjecture the right bound is some $\tilde{O}(d/\epsilon)$

An example in the plane

Given: μ is the uniform distribution on m points, $\epsilon > 0$

$\epsilon \cdot m = 6$



Objective: find a small ϵ-net w.r.t μ.

(hit every convex set with at least 6 blue points)

What properties of convex sets enable weak epsilon nets?

- Proofs use a lot of geometry
- Goal: identify a simple/basic property of convex sets that captures existence of weak epsilon nets
- notion of dimension?
- It is convenient to consider the framework of ***abstract convexity*** [Levi '51, Danzer-Grünbaum-Klee '63, Kay-Womble '71]

Abstract *convexity spaces*

Definition. A *convexity space* is a pair *(X,F)* where *F* is a family of subsets of *X* that is closed under intersections.

Some convexity spaces:

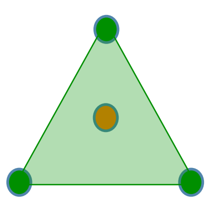
- closed/compact sets
- subgroups/subfields/subrings
- euclidean convex sets, grid convex sets
- geodesic convex sets
- …

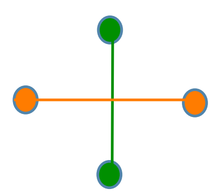
[*"Math is the art of giving the same name to different things"* – Henri Poincaré]

Definition. *(X,F)* has *weak ϵ-nets* if there is n=n(ϵ) s.t: for every distribution μ there are n points that hit all sets in *F* with measure at least ϵ.

The Radon number of abstract convexity spaces

Theorem. *[Radon '21]*
Any *d+2* points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect.





Definition. The ***Radon number*** of *(X,F)* is the smallest integer *r* s.t any set of *r* points can be partitioned into two sets whose *convex hulls* intersect.

Conjecture: *Radon number* captures weak epsilon nets

Conjecture.
Let *(X,F)* be a convexity space. The following are equivalent
1. *(X,F)* has a finite radon number.
2. *(X,F)* has a weak ϵ-nets of for every $\epsilon > 0$.

2 → 1 is known [M-Yehudayoff '17].

- proof by a simple reduction to the *chromatic number of **Kneser graphs***
  - analyzing the chromatic number of Kneser graphs [Lovász 1978] is one of the first applications of the **topological method** in combinatorics.

Theorem: *Radon number* captures weak epsilon nets for separable spaces

Theorem. [M-Yehudayoff '17]
Let *(X,F)* be a *separable* convexity space.
The following are equivalent

1. *(X,F)* has a finite radon number.
2. *(X,F)* has a weak ϵ-nets of finite size for every $\epsilon > 0$.

Definition. Let *(X,F)* be a convexity space. A *hyperplane* is a partition of *X* into two convex sets (members of *F*).

Definition. *(X,F)* is *separable* if for every $C \in F$ and $x \notin C$ there is a *hyperplane* separating *x* and *C*. [abstraction of Hahn-Banach]

Remains open: does a finite Radon number imply weak epsilon nets in non-separable spaces?

Example. (non-separable convexity space)

Let *G* be a group with identity *e.*

Consider the convexity space *(G,F)*, where

$$F = \{H \setminus \{e\} : H \leq G\}.$$

Radon number *r* means:
any *r* elements can be partitioned to two sets A,B whose generated groups *<A>, <B>* share an element $x \neq e$.

Thank you!