ON THE MATHEMATICAL THEORY OF BLACK HOLES

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October 18, 2017

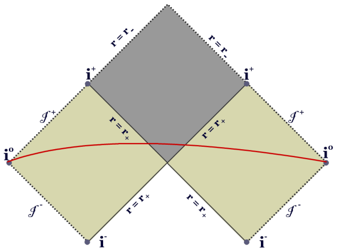
THIRD LECTURE

1. QUICK REVIEW
2. MAIN RECENT ADVANCES
3. AXIAL SYMMETRIC POLARIZED SPACETIME
4. MAIN RESULT
5. MAIN FEATURES OF THE CONSTRUCTION
6. MAIN FEATURES OF THE PROOF

STABILITY

CONJECTURE[Stability of (external) Kerr].

Small perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



GENERAL STABILITY PROBLEM $\mathcal{N}[\phi_0] = 0.$

NONLINEAR EQUATIONS. $\mathcal{N}[\phi_0 + \psi] = 0$

1. ORBITAL STABILITY(OS). ψ bounded for all time.
2. ASYMPT STABILITY(AS). $\psi \longrightarrow 0$ as $t \to \infty$.

LINEARIZED EQUATIONS. $\mathcal{N}'[\phi_0]\psi = 0.$

1. MODE STABILITY (MS). No growing modes.
2. BOUNDEDNESS.
3. QUANTITATIVE DECAY.

GENERAL STABILITY PROBLEM $\mathcal{N}[\phi_0] = 0$

STATIONARY CASE. Expect linear instabilities due to non-decaying states in the kernel $\mathcal{N}'[\phi_0]$. **Due to**

1. Presence of continuous family of stationary solutions. ϕ_λ
   Implies that the final state ϕ_f may differ from initial state ϕ_0

2. Presence of a continuous family of invariant diffeomorphisms. Requires us to track dynamically the gauge condition to insure decay of solutions towards the final state.

QUANTITATIVE LINEAR STABILITY. After accounting for (1) and (2), all solutions of $\mathcal{N}'[\phi_0]\psi = 0$ decay sufficiently fast.

MODULATION. Method of constructing solutions to the nonlinear problem by tracking (1) and (2).

GEOMETRIC FRAMEWORK FOR STABILITY

1. Principal Null Directions e_3, e_4.
2. Horizontal Structure. Null Frames.
3. Null decompositions
   - Connection $\Gamma = \{\chi, \xi, \eta, \zeta, \underline{\eta}, \omega, \underline{\xi}, \underline{\omega}\}$
   - Curvature $R = \{\alpha, \beta, \rho, {}^\star\rho, \underline{\beta}, \underline{\alpha}\}$
4. $O(\epsilon)$ -Perturbations
5. $O(\epsilon)$ -Frame Transformations. Invariant quantities.
6. Main Equations

$O(\epsilon)$ - PERTURBATIONS OF KERR

ASSUME. There exists an null frame e_3, e_4, e_1, e_2 such that

$$\xi,\ \underline{\xi},\ \widehat{\chi},\ \underline{\widehat{\chi}},\ \alpha,\ \underline{\alpha},\ \beta,\ \underline{\beta} = O(\epsilon)$$

FRAME TRANSFORMATIONS, $(f_a)_{a=1,2},\ (\underline{f}_a)_{a=1,2} = O(\epsilon)$

$$\begin{aligned} e_4' &= e_4 + f_a e_a + O(\epsilon^2) \\ e_3' &= e_3 + \underline{f}_a e_a + O(\epsilon^2) \\ e_a' &= e_a + \frac{1}{2}\underline{f}_a e_4 + \frac{1}{2} f_a e_3 + O(\epsilon^2) \end{aligned}$$

- The curvature components $\alpha, \underline{\alpha}$ are $O(\epsilon^2)$ invariant with respect to $O(\epsilon)-$ gauge transformations
- For $O(\epsilon)$-perturbations of Minkowski all null components of R are $O(\epsilon^2)$-invariant.

BASIC EQUATIONS

NULL STRUCTURE EQTS. (Transport)

$$\begin{aligned} \nabla_4 \Gamma &= \mathbf{R} + \Gamma \cdot \Gamma, \\ \nabla_3 \Gamma &= \mathbf{R} + \Gamma \cdot \Gamma \end{aligned}$$

NULL STRUCTURE EQTS. (Codazzi)

$$\nabla \Gamma = \mathbf{R} + \Gamma \cdot \Gamma$$

NULL BIANCHI.

$$\nabla_4 \mathbf{R} = \nabla \mathbf{R} + \Gamma \cdot \mathbf{R}, \quad \nabla_3 \mathbf{R} = \nabla \mathbf{R} + \Gamma \cdot \mathbf{R}$$

KERR STABILITY-MAIN DIFFICULTIES

UNLIKE STABILITY OF MINKOWSKI

1. Some null curvature components (middle components) are nontrivial. Bianchi system admits **non-decaying** states.

2. The null decomposition of the curvature tensor is **sensitive** to frame transformations.

3. Principal null directions are **not integrable**.

4. Have to track the parameters (a_f, m_f) of the final Ker and the correct gauge condition. **Have to emerge dynamically !**

5. Obstacles to prove decay for the simplest linear equations $\square_{\mathbf{g}} \Phi = 0$ on a fixed Kerr.

MAIN RECENT ADVANCES

1. TEUKOLSKI EQTS.
2. CHANDRASKHAR TRANSFORMATION
3. CLASSICAL AND NEW VF. METHOD
4. LINEAR STABILITY OF SCHWARZSCHILD

SUMMARY

WHAT WE UNDERSTAND. In light of the recent advances we now have tools to control, in principle, $\alpha, \underline{\alpha}$. This replaces the methods used in the stability of Minkowski based on the analysis of the Bianchi system.

WHAT REMAINS TO DO.

- Find quantities that track the mass and angular momentum.
- Find an effective, **dynamical method** to fix the gauge problem.
- Determine the decay properties of all important quantities and close the estimates for the full nonlinear problem.

AXIAL SYMMETRIC POLARIZED SPACETIMES

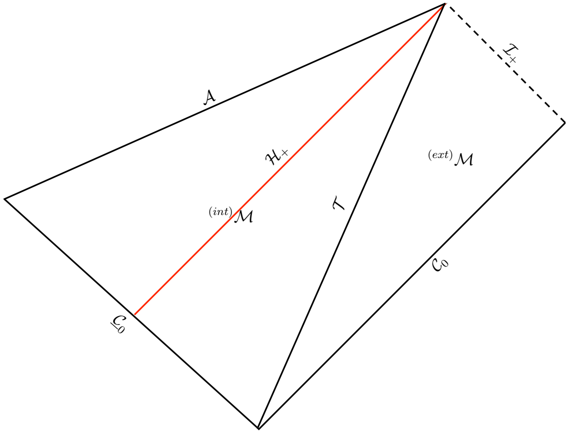
$$\mathbf{g} = e^{2\Phi} d\varphi^2 + g_{ab} dx^a dx^b$$

SIMPLIFICATIONS.

- Final State must be Schwarzschild.
- Principal null directions are integrable in Schwarzschild. We can use a geometric description based on optical functions. Geodesic foliations.
- Hawking mass is a good candidate to track the final mass.
- We control in principle! the extreme components $\alpha, \underline{\alpha}$.

AXIAL SYMMETRIC POLARIZED SPACETIMES

THEOREM[K-Szeftel] Small axial polarized perturbations of given initial conditions of an exterior Schwarzschild $\mathbf{g}_{m_0}$ $(m_0 > 0)$ have maximal future developments converging to another exterior Schw. solution $\mathbf{g}_{m_\infty}$, $m_\infty > 0$.



KEY FEATURES OF THE CONSTRUCTION

(1) Use optical functions $u, \underline{u}$ initialized on $\mathcal{T}$.

(2) The timelike hypersurface $\mathcal{T}$ is foliated by a special class of 2-surfaces; generally covariant modulated spheres GCMS.

(3) GCMS makes use of the full number of degrees of freedom of the diffeomorphism group to set to zero three key quantities

$$\text{tr}\chi - \overline{\text{tr}\chi} = \text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}} = \mu - \overline{\mu} = 0.$$

(4) The GCMS foliation on $\mathcal{T}$ defines

- An outgoing geodesic foliation in ${}^{(int)}\mathcal{M}$ - Optical function u.
- An ingoing geodesic foliation in ${}^{(int)}\mathcal{M}$ - Optical function $\underline{u}$.
- Null frames in ${}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$.

(5) Together with the knowledge of $\alpha, \underline{\alpha}$ the GCMS determine all other connection and curvature components on $\mathcal{T}$.

KEY FEATURES OF THE CONSTRUCTION

(6) Hawking mass $\frac{2m_H(u,r)}{r} = 1 + \frac{1}{16\pi}\int_S \mathrm{tr}\chi\,\mathrm{tr}\underline{\chi}$.

(7) The final mass is determined, in principle, by

$$m_\infty = \lim_{u\to\infty}\lim_{r\to\infty} m_H(u,r).$$

(8) All connection and curvature components are determined by transport equations from their initial values on $\mathcal{T}$ and $\alpha, \underline{\alpha}$.

(9) The spacetime $\mathcal{M}$, timelike hypersurface $\mathcal{T}$ and the two geodesic foliations are constructed by a continuity argument starting with an initial data layer $\mathcal{L}_0 \cup \underline{\mathcal{L}}_0$.

(10) GSMS admissible spacetimes.

COMPLETE STATEMENT OF THEOREM

INITIAL LAYER ASSUMPTION. $\mathcal{I}_{k_{large}+5} \leq \epsilon_0$

CONCLUSIONS. There exists a future globally hyperbolic GCMS development with complete future null infinity $\mathcal{I}_+$ and future horizon $\mathcal{H}_+$ which verifies

$$\mathcal{N}^{(En)}_{k_{large}} + \mathcal{N}^{(Dec)}_{k_{small}} \leq C\epsilon_0, \qquad k_{small} = \left\lfloor \frac{1}{2} k_{large} \right\rfloor + 1.$$

In particular,

- On ${}^{(ext)}\mathcal{M}$, we have,

$$|\alpha|, |\beta| \lesssim \epsilon_0 \min\left\{ \frac{1}{r^3(u+2r)^{\frac{1}{2}+\delta_{dec}}}, \frac{1}{r^2(u+2r)^{1+\delta_{dec}}} \right\},$$

$$|\underline{\beta}| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}, \qquad |\underline{\alpha}| \lesssim \frac{\epsilon_0}{r u^{1+\delta_{dec}}},$$

$$|\widehat{\chi}|, |\zeta| \lesssim \epsilon_0 \min\left\{ \frac{1}{r^2 u^{\frac{1}{2}+\delta_{dec}}}, \frac{1}{r u^{1+\delta_{dec}}} \right\}, \qquad |\underline{\widehat{\chi}}| \lesssim \frac{\epsilon_0}{r u^{1+\delta_{dec}}}.$$

- On ${}^{(int)}\mathcal{M}$,

$$|\alpha|, |\beta|, |\underline{\beta}|, |\underline{\alpha}|, |\widehat{\chi}|, |\zeta|, |\widehat{\underline{\chi}}| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}.$$

- $m_\infty = \lim_{u\to\infty} \lim_{r\to\infty} m_H(u, r), \quad |m_\infty - m_0| \lesssim \epsilon_0.$

- On the future Horizon $\mathcal{H}_+$,

$$r = 2m_\infty + O\left(\frac{\sqrt{\epsilon_0}}{\underline{u}^{1+\frac{\delta_{dec}}{2}}}\right) \text{ on } \mathcal{H}_+$$

- On ${}^{(ext)}\mathcal{M}$,

$$\left|\rho + \frac{2m_\infty}{r^3}\right| \lesssim \epsilon_0 \min\{\frac{1}{r^2 u^{1+\delta_{dec}}}, \frac{1}{r^3 u^{1/2+\delta_{dec}}}\}$$

$$\left|\mathrm{tr}\chi - \frac{2}{r}\right| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta_{dec}}}, \quad \left|\mathrm{tr}\underline{\chi} + \frac{2\left(1 - \frac{2m_\infty}{r}\right)}{r}\right| \lesssim \frac{\epsilon_0}{r u^{1+\delta_{dec}}}.$$

- On ${}^{(int)}\mathcal{M}$, we have

$$\left|\rho+\frac{2m_\infty}{r^3}\right|, \left|\underline{\kappa}+\frac{2}{r}\right|, \left|\kappa-\frac{2\left(1-\frac{2m_\infty}{r}\right)}{r}\right| \lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}.$$

- On ${}^{(ext)}\mathcal{M}$, in u, r, θ, φ coordinates

$$\begin{aligned}\mathbf{g} &= \mathbf{g}_{m_\infty, {}^{(ext)}\mathcal{M}} + O\left(\frac{\epsilon_0}{u^{1+\delta_{dec}}}\right)\\ \mathbf{g}_{m_\infty, {}^{(ext)}\mathcal{M}} &= -2dudr - \left(1-\frac{2m_\infty}{r}\right)(du)^2 + r^2 d\sigma^2.\end{aligned}$$

- On ${}^{(int)}\mathcal{M}$, in $\underline{u}, r, \theta, \varphi$ coordinates

$$\begin{aligned}\mathbf{g} &= \mathbf{g}_{m_\infty, {}^{(int)}\mathcal{M}} + O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta_{dec}}}\right)\\ \mathbf{g}_{m_\infty, {}^{(int)}\mathcal{M}} &= 2d\underline{u}dr - \left(1-\frac{2m_\infty}{r}\right)(d\underline{u})^2 + r^2 d\sigma^2\end{aligned}$$

OTHER CONCLUSIONS

BONDI MASS. $M_B(u) = \lim_{r\to+\infty} m(u,r)$ for all $0 \le u < +\infty$

BONDI MASS LAW FORMULA.

$$\partial_u M_B(u) \quad = \quad -\frac{1}{16}\int_{S_\infty(u)} \underline{\Theta}^2(u,\cdot) \text{ for all } 0 \le u < +\infty.$$

with

$$\underline{\Theta}(u,\cdot) \quad = \quad \lim_{r\to+\infty} r\widehat{\underline{\chi}}(r,u,\cdot) \text{ for all } 0 \le u < +\infty.$$

FINAL BONDI MASS.

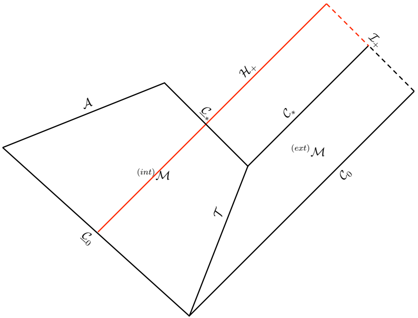
$$M_B(+\infty) = \lim_{u\to+\infty} M_B(u) = m_\infty$$

MAIN INTERMEDIATE STEPS

THM 0. There exists a **GCMS** $S_0 \subset \mathcal{L}_0$, $r = 2m_0(1+\delta)$ generating $\underline{\mathcal{C}}_0 \cup \mathcal{C}_0$,

$$\mathcal{N}^{(En)}_{k_{large}}(0) + \sup_{\mathcal{C}_0 \cup \underline{\mathcal{C}}_0} |m - m_0| \lesssim \epsilon_0$$

BOOTSTRAP ASSUMPTION (BA).



$$\mathcal{N}^{(En)}_{k_{large}}, \ \mathcal{N}^{(Dec)}_{k_{small}} \leq \epsilon$$

MAIN INTERMEDIATE STEPS

THM 1. Given a GCM admissible spacetime verifying $\mathcal{N}^{(En)}_{k_{large}}(0) \lesssim \epsilon_0$ and **BA**, we deduce, for $\square \mathfrak{q} + V\mathfrak{q} = \text{Err}$,

$$\mathcal{N}^{(dec)}_{k_{small}+20}[\mathfrak{q}] \lesssim \epsilon_0.$$

THM 2-3. Under the same assumptions we have in ${}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$,

$$\mathcal{N}^{(dec)}_{k_{small}+15}[\alpha, \underline{\alpha}] \lesssim \epsilon_0.$$

THM 4, 5. Under the same assumptions, we have in ${}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$,

$$\mathcal{N}^{(dec)}_{k_{small}+5}[\check{R}, \check{\Gamma}] \lesssim \epsilon_0.$$

THM 6. Under the same assumptions as above we have in $\mathcal{M}$,

$$\mathcal{N}^{(En)}_{k_{large}} + \mathcal{N}^{(Dec)}_{k_{small}+5} \lesssim \epsilon_0.$$

MAIN INTERMEDIATE STEPS

DEFINITION. Let $\mathcal{U} \subset \mathbb{R}_+$ the set all values of u_* such an admissible spacetime exists for $u \in [0, u_*]$ verifying **BA**.

THM 7. There exists $\delta_0 > 0$ small enough such that for sufficiently small $\epsilon_0 > 0$, $\epsilon > 0$, $[0, \delta_0] \subset \mathcal{U}$.

THM 8. Given a GCM admissible spacetime with $0 < u_* < +\infty$ such that

$$\mathcal{N}^{(En)}_{k_{large}} + \mathcal{N}^{(Dec)}_{k_{small}} \lesssim \epsilon_0,$$

we can find $u'_* > u_*$ such that $u'_* \in \mathcal{U}$.

CONSTRUCTION OF GCMS

METRIC.

$$\mathbf{g} = -2duds + \underline{\Omega} du^2 + \gamma \left(d\theta - \frac{1}{2} b du \right)^2 + e^{2\Phi} d\varphi^2.$$

FRAME TRANSFORMATIONS. $f, \underline{f}, a = O(\epsilon)$,

$$\begin{aligned} e_3' &= e^a (e_3 + \underline{f} e_\theta + \frac{1}{4} \underline{f}^2 e_4) \\ e_\theta' &= (1 + \frac{1}{2} f \underline{f}) e_\theta + \frac{1}{2} (f e_3 + \underline{f} e_4) + \text{l.o.t.} \\ e_4' &= e^{-a} \left((1 + \frac{1}{2} f \underline{f}) e_4 + f e_\theta + \frac{1}{4} f^2 e_3 \right) + \text{l.o.t.} \end{aligned}$$

DEFORMATIONS. $\Psi : \overset{\circ}{\mathbf{S}} \longrightarrow \mathbf{S}$

$$u = \overset{\circ}{u} + U(\theta), \quad s = \overset{\circ}{s} + S(\theta), \qquad \theta \in [0, \pi].$$

CONSTRUCTION OF GCMS

PROPOSITION. Given $\overset{\circ}{\mathbf{S}}$ with $\overset{\circ}{r} = 2m_0(1+\delta_{\mathcal{H}})$ there exists a nearby deformed sphere $\mathbf{S}$ of area radius $r^{\mathbf{S}} = \overset{\circ}{r} + O(\epsilon)$, and a compatible frame

$$(e_3' = e_3^S, e_4' = e_4^{\mathbf{S}}, e_\theta' = e_\theta^{\mathbf{S}})$$

which verifies the GCM conditions

$$\kappa^S = \frac{2}{r^S}, \qquad \underline{\check{\kappa}}^S = \check{\mu}^S = 0.$$

CONSTRUCTION OF GCMS

ADAPTED FRAME TRANSFORMATIONS

$$\Psi_*(e_\theta) = e_\theta^{\mathbf{S}}$$

COMPATIBILITY. $U, S : [0, \pi] \longrightarrow [0, \pi]$, $U(0) = S(0) = 0$, uniquely determined in terms of a, f, $\underline{f}$ by transport type equations.

GCMS- CONDITION. Leads to a nonlinear elliptic Hodge system on $\mathbf{S}$ for $a, f, \underline{f}$ which has trivial kernel if $\overset{\circ}{\mathbf{S}}$ is sufficiently close to $r = 2m_0$.

CONSTRUCTION OF GCMS

ITERATION. Define iteratively quintets $Q^{(n)} = (U^{(n)}, S^{(n)}, a^{(n)}, f^{(n)}, \underline{f}^{(n)})$ starting with $Q^{(0)}$.

- $Q^{(0)}$ represents the trivial deformation.
- $(U^{(n)}, S^{(n)})$ defines the map $\Psi^{(n)} : \overset{\circ}{\mathbf{S}} \longrightarrow \mathbf{S}(n)$. Define the triplet $(a^{(n+1)}, f^{(n+1)}, \underline{f}^{(n+1)})$ by solving the nonlinear elliptic system on $\mathbf{S}(n)$,

$$\mathcal{D}^{(n)}(f^{(n+1)}, \underline{f}^{(n+1)}, a^{(n+1)}) = 0$$

- Construct the pair $U^{(n+1)}, S^{(n+1)}$by solving a transport equation defined by the triplet $(a^{(n+1)}, f^{(n+1)}, \underline{f}^{(n+1)})$.
- Contraction Argument. Need to compare the pull backs to $\overset{\circ}{\mathbf{S}}$.