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- The group operation $\ast$ of the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $+$ is the group operation of $\mathbb{R}^2$ and $\mathbb{R}$, is given by

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- The intrinsically flat horizontal planes $\mathbb{R}^2 \rtimes_A \{t\}$ in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ have constant mean curvature $\text{trace}(A)/2$. 
Notation and language:

- $Y$ denotes a simply connected 3-dimensional homogeneous manifold.
- $H(Y) = \inf\{\max |H_M| : M = \text{immersed closed surface in } Y\}$,
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- If $Y$ is diffeomorphic to $S^3$, then $H(Y) = 0$ since there exist closed minimal surfaces in such a space $Y$.
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- In particular, $H(Y) = 1$ if $Y = \mathbb{H}^3$ and $H(Y) = 1/2$ if $Y = \mathbb{H}^2 \times \mathbb{R}$.  

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The proof of

\[ 2H(Y) = \inf_{K \subset Y \text{ compact}} \frac{\text{Area}(\partial K)}{\text{Volume}(K)} = \text{Cheeger constant of } Y \]

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1. If \( \Omega(n) \subset Y \) is a sequence of isoperimetric domains in \( Y \) with \( \text{Volume}(\Omega(n)) \to \infty \), then:

\[
\lim_{n \to \infty} \text{Radius}(\Omega(n)) = \infty \implies k \gg 0, \quad 2H(Y) < H(\partial \Omega(n)) < H(\partial \Omega(n+1)).
\]

\[
\lim_{n \to \infty} H(\partial \Omega(n)) = H(Y). \quad \text{(Study the Isoperimetric Profile \( P \) of \( Y \).)}
\]

\[
2 \cdot H(\partial \Omega(n)) = \text{Ch}(Y). \quad \text{(Prove \( P \) has asymptotic slope \( \text{Ch}(Y) \).)}
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2. In this case where \( Y \) is diffeomorphic to \( \mathbb{R}^3 \), the leaves of the foliation \( \mathcal{F} \) of \( Y \) are invariant under a \( 1 \)-parameter group of isometries of \( Y \). By the maximum principle, there are no closed immersed \( H(Y) \)-surfaces in \( Y \).
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Let $M$ be a homogeneous 3-manifold, $X$ denote its Riemannian universal cover, $Ch(X)$ denote the Cheeger constant of $X$.

The next theorem solves the so called **Hopf Uniqueness Problem**.
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**Theorem (Hopf Uniqueness Problem, 2017 Meeks-Mira-Pérez-Ros)**

Any two spheres in $\mathbf{M}$ of the same absolute constant mean curvature differ by an isometry of $\mathbf{M}$.
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Any two spheres in $M$ of the same absolute constant mean curvature differ by an isometry of $M$. Moreover:

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Embedded constant mean curvature surfaces
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Theorem (Geometry of $H$-spheres, 2017 Meeks-Mira-Pérez-Ros)

Let $S$ be an $H$-sphere in $M$.

1. If $H = 0$ and $X$ is a product $S^2 \times \mathbb{R}$, where $S^2$ is a sphere of constant curvature, then $S$ is totally geodesic, stable and has nullity 1 for its Jacobi operator.
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3. There is a point $p_S \in M$, called the center of symmetry of $S$, such that every isometry of $M$ that fixes $p_S$ also leaves invariant $S$. 

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Embedded constant mean curvature surfaces
Previous influential results on the **Hopf Uniqueness Problem**:

**Theorem (Hopf, 1951)**

**H-spheres in** $\mathbb{R}^3$ **are round.**

**Theorem (Abresch-Rosenberg, 2004)**

If $M$ has a 4-dimensional isometry group, then **H-spheres in** $M$ **are surfaces of revolution and they are unique.**

**Theorem (Daniel-Mira (2013), Meeks (2013))**

If $X$ is the Lie group $\text{Sol}_3$ with the left invariant metric $e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$, then **H-spheres in** $X$ **are unique, embedded and have index 1.** After left translation, these spheres have ambient symmetry group generated by reflections in the ($x,z$) and ($y,z$)-planes and rotations by $\pi$ around the two lines $y = \pm x$ in the ($x,y$)-plane.
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Theorem (Classification Theorem for $H$-spheres, Meeks-Mira-Pérez-Ros)

Suppose $X$ is a simply connected 3-dimensional homogeneous manifold different from $S^2(\kappa) \times \mathbb{R}$, where $S^2(\kappa)$ is a sphere of curvature $\kappa$.

- $X$ diffeomorphic to $S^3 \implies$ the moduli space of $H$-spheres in $X$ is parameterized by the mean curvature values $H \in \mathbb{R}$.

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Remark

In the following proof, choose a metric Lie group structure on $X$. 

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Embedded constant mean curvature surfaces
Definition (Left invariant Gauss map)

- Let $X$ be a 3-dimensional metric Lie group.
- Given an oriented immersed surface $f : M \to X$ with unit normal vector field $\xi$, the left invariant Gauss map of $M$ is the map $G : M \to S^2 \subset T_eX$ that assigns to each $p \in M$, the unit tangent vector to $X$ at the identity element $e$ given by left translation:

$$ (dl_{f(p)})_e(G(p)) = \xi_p. $$
Steps of the proof of the Classification Theorem for $H$-spheres.

Throughout $\Sigma$ denotes a fixed $H_0$-sphere in $X$ of index 1.
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- **Step 4:** Area estimates for $\Sigma$. 

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Embedded constant mean curvature surfaces
Steps of the proof of the Classification Theorem for $H$-spheres.

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- **Step 0:** $\Sigma$ has nullity 3: Cheng’s theorem.
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- **Step 2:** The left invariant Gauss map $G: \Sigma \to S^2 \subset T_e(X)$ is a degree-1 diffeomorphism: Nodal Domain Argument + Rep Thm.
- **Step 3:** Curvature estimates for $\Sigma$ (given any fixed upper bound $H_1$ of $H_0$): Use that Gauss map is a degree-1 diffeo.
- **Step 4:** Area estimates for $\Sigma$. This means:
  
  (A) If $X$ is isomorphic to $SU(2)$, areas of spheres in $\mathcal{M}(X)$ are uniformly bounded.
Steps of the proof of the Classification Theorem for $H$-spheres.

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Recall, there are no $H(X)$-spheres in $X$. 

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Embedded constant mean curvature surfaces
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Embedded constant mean curvature surfaces
Steps of the proof continued.

- **Step 5**: Components of $\mathcal{M}(X)$ are parameterized by the mean curvature values $[0, \infty)$ if $X$ is isomorphic to $\text{SU}(2)$ and otherwise by $(H(X), \infty)$.
Steps of the proof continued.

- **Step 5:** Components of $\mathcal{M}(X)$ are parameterized by the mean curvature values $[0, \infty)$ if $X$ is isomorphic to $SU(2)$ and otherwise by $(H(X), \infty)$.

- **Step 6:** On any $H_0$-sphere $M$ different from a left translation of $\Sigma$, there exists a **NON-ZERO** complex valued quadratic differential $\omega_\Sigma(M)$ with isolated negative index zeroes. **Depends on Representation Thm.**

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**Conclusions:**
- The space of non-congruent $H$-spheres in $X$ equals $\mathcal{M}(X)$ which is an interval parameterized by the mean curvature values in $[0, \infty)$ if $X$ is isomorphic to $SU(2)$ and otherwise, in the interval $(H(X), \infty)$.
- Each $H$-sphere in $X$ has **index 1** and **nullity 3**.
- Each $H$-sphere in $X$ is the boundary of an immersed 3-ball $F: B \to X$ on its mean convex side (Alexandrov embedded).
- If $X$ is isomorphic to $SU(2)$, then the areas of $H$-spheres in $X$ form a half-open interval $(0, A(X)]$. 

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*Embedded constant mean curvature surfaces*
Theorem (Curvature Estimates for $H$-Disks, Meeks-Tinaglia 2018)

Fix $\varepsilon, H_0 > 0$ and a complete locally homogenous 3-manifold $X$. $\exists C > 0$ s.t. for all embedded $(H \geq H_0)$-disks $D$ in $X$:

$$|A_D|(p) \leq C \text{ for all } p \in D \text{ s.t. } \text{dist}_D(p, \partial D) \geq \varepsilon,$$

where $|A_D|$ denotes the norm of second fundamental form.

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- **Step 3:** But the limit must have 0 CMC flux $\implies D'$ is a helicoid.
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**Step 4:** One extends the double multigraph in the forming helicoid near $p_n \in D(n)$ a definite distance for $n$ large, a contradiction.
Theorem (One-sided curvature estimate for $\textbf{H}$-disks, Meeks-Tinaglia)

$\exists C, \varepsilon > 0$ s.t. for any embedded $\textbf{H}$-disk $\Sigma \subset \mathbb{R}^3$ as in the figure below:

$$|A_{\Sigma}| \leq \frac{C}{R} \text{ in } \Sigma \cap B(\varepsilon R) \cap \{x_3 > 0\}.$$

This result generalizes the one-sided curvature estimates for minimal disks by Colding-Minicozzi, and uses their work in its proof.
New uniqueness results for CMC surfaces.

Old Question
Is the round sphere the only complete simply connected surface embedded in $\mathbb{R}^3$ with non-zero constant mean curvature?

Answer (Meeks-Tinaglia) Yes!

NOT simply connected
- Cylinder

NOT embedded
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Corollary (Radius Estimates for $\mathbf{H}$-Disks, Meeks-Tinaglia 2017)

$\exists R_0 \geq \pi$ such that every embedded $1$-disk in $\mathbb{R}^3$ has radius $< R_0$. 

Proof. Let $D(n) \subset \mathbb{R}^3$ be a sequence of embedded $1$-disks of radius $R(n) > n$. The homothetically scaled disks $D(n) = \frac{1}{R(n)} D(n)$ contain points $p_n$ of distance $1$ from the boundary with mean curvature $R(n) > n$. So, $|A_{D(n)}(p_n)| > n$, which contradicts the curvature estimates for $(R(n) \geq 1)$-disks with $\varepsilon = 1$. 

Corollary (Meeks-Tinaglia 2017)

A complete simply connected $\mathbf{H}$-surface embedded in $\mathbb{R}^3$ with $H > 0$ is a round sphere.

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Embedded constant mean curvature surfaces
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Let $M \subset \mathbb{R}^3$ be a complete, connected embedded $H$-surface.
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Let $M \subset \mathbb{R}^3$ be a complete, connected embedded $H$-surface.

1. $M$ has positive injectivity radius $\Rightarrow$ $M$ is properly embedded in $\mathbb{R}^3$. 

2. $M$ has finite topology $\Rightarrow M$ has positive injectivity radius.

3. Suppose $H > 0$. Then: $|A_M|$ is bounded $\iff M$ has positive injectivity radius.

When $H = 0$, items 1 and 2 were proved by Meeks-Rosenberg, based on: Colding-Minicozzi: $M$ has finite topology and $H = 0 \Rightarrow M$ is proper.

Item 3 in the above theorem holds for 3-manifolds which have bounded absolute sectional curvature; in particular it holds in closed Riemannian 3-manifolds.
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Universal domain for Embedded Calabi-Yau problem?

- $\mathcal{D}_\infty =$ above **bounded domain, smooth except at $p_\infty$ on right.**
- **Ferrer, Martin and Meeks** conjecture: An open surface **properly embeds as a complete minimal surface in $\mathcal{D}_\infty$** $\iff$ every end has **infinite genus** $\iff$ it admits a complete bounded minimal embedding in $\mathbb{R}^3$. 

Embedded constant mean curvature surfaces
Conjecture (General Calabi-Yau Conjecture, Meeks-Pérez-Ros-Tinaglia)

Let $\Sigma \subset \mathbb{R}^3$ be a complete, connected embedded $H$-surface.

The General Calabi-Yau Conjecture is true if and only if $\Sigma$ has a countable number of ends if and only if $\Sigma$ has at most 2 limit ends.
Conjecture (General Calabi-Yau Conjecture, Meeks-Pérez-Ros-Tinaglia)

Let $\Sigma \subset \mathbb{R}^3$ be a complete, connected embedded $H$-surface.

- $\Sigma$ is an $H$-lamination in $\mathbb{R}^3$ iff $\Sigma$ has locally bounded genus in $\mathbb{R}^3$. 

Theorem (Emb Calabi-Yau for Finite Genus, Meeks-Pérez-Ros (2017))

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Let $\Sigma \subset R^3$ be a complete, connected embedded $H$-surface.

- $\overline{\Sigma}$ is an $H$-lamination in $R^3$ iff $\Sigma$ has locally bounded genus in $R^3$.

- $\exists A_\Sigma$ s.t. $\forall r \geq 1$ and $p \in R^3$,

$$\text{Area}(\Sigma \cap B(p, r)) \leq A_\Sigma \cdot r^3$$

iff $\Sigma$ has uniformly bounded genus in balls of radius 1.

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Next theorem is motivated by the study of 3-periodic $H$-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $g > 2$ in any flat 3-torus (Traizet).
Figure: A body-centered cubic interface or Fermi surface in salt crystal.

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**Theorem (Meeks-Tinaglia(2016))**

Given a flat 3-torus $\mathbb{T}^3$ and $H > 0$, $\forall g \in \mathbb{N}$, $\exists C(g, H)$ s.t. a closed $H$-surface $\Sigma$ embedded in $\mathbb{T}^3$ with genus at most $g$ satisfies

$$\text{Area}(\Sigma) \leq C(g, H).$$
Closed H-surfaces in a flat 3-torus. By K. Grosse-Brauckmann (top) and N. Schmitt (bottom)

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Embedded constant mean curvature surfaces
Theorem (Choi-Wang(1983), Choi-Schoen(1985))

Let $N$ = a closed Riemannian 3-manifold with Ricci curvature $> 0$. Then:

1. The areas of closed, connected embedded minimal surfaces of fixed genus in $N$ are bounded

2. The space of embedded closed minimal surfaces of fixed genus in $N$ is compact.

Theorem (Meeks-Tinaglia(2018))

Let $0 < a \leq b$ and $N$ = closed Riem. 3-manifold with $H^2(N) = 0$. Then:

1. The areas and indexes of stability of closed, connected embedded $H$-surfaces of fixed genus $g$ in $N$ with $H \in [a, b]$ are uniformly bounded.

2. For every closed Riemannian 3-manifold $X$ and any non-negative integer $g$, the space of strongly Alexandrov embedded closed surfaces in $X$ of genus at most $g$ and constant mean curvature $H \in [a, b]$ is compact. (Similar compactness result holds for any fixed smooth compact family of metrics on $X$.)
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1. The areas and indexes of stability of closed, connected embedded $H$-surfaces of fixed genus $g$ in $N$ with $H \in [a, b]$ are bounded are uniformly bounded.
Theorem (Choi-Wang(1983), Choi-Schoen(1985))

Let \( N \) = a closed Riemannian 3-manifold with Ricci curvature \( > 0 \).
Then:

1. The areas of closed, connected embedded minimal surfaces of fixed genus in \( N \) are bounded.
2. The space of embedded closed minimal surfaces of fixed genus in \( N \) is compact.

Theorem (Meeks-Tinaglia(2018))

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Then:

1. The areas and indexes of stability of closed, connected embedded \( H \)-surfaces of fixed genus \( g \) in \( N \) with \( H \in [a, b] \) are uniformly bounded.
2. For every closed Riemannian 3-manifold \( X \) and any non-negative integer \( g \), the space of strongly Alexandrov embedded closed surfaces in \( X \) of genus at most \( g \) and constant mean curvature \( H \in [a, b] \) is compact. (Similar compactness result holds for any fixed smooth compact family of metrics on \( X \).)
Theorem (Meeks-Tinaglia (2018))

- For $H \geq 1$, complete embedded finite topology $H$-surfaces $\Sigma$ in complete hyperbolic 3-manifolds are proper.
Calabi-Yau type problems for embedded $H$-surfaces

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- In particular, by results of Collin, Hauswirth, Rosenberg in the case $H = 1$ and of Korevaar, Kusner, Meeks, Solomon in the case $H > 1$, $\Sigma$ has ends asymptotic to annuli of revolution.

Theorem (Coskunuzer-Meeks-Tinaglia (2017))

- For every $H \in [0, 1)$, $\exists$ a complete embedded stable $H$-plane that is nonproper in the hyperbolic 3-space $H^3$.

- For every $H \in (0, 1/2)$, $\exists$ a complete embedded stable $H$-plane that is nonproper in the Riemannian product $H^2 \times \mathbb{R}$.

Theorem (Tinaglia-Rodriguez)

- $\exists$ a complete embedded stable minimal plane that is nonproper in $H^2 \times \mathbb{R}$. 

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$\exists$ a complete embedded stable minimal plane that is **nonproper** in $\mathbb{H}^2 \times \mathbb{R}$.
Suppose $X$ is a complete hyperbolic 3-manifold with finite volume, $H \in [0, 1)$ and $M$ is a properly immersed $H$-surface. Then:

- $M$ has finite area and total curvature $2\pi \chi(M)$.
- $M$ has bounded fundamental form $\iff$ $M$ has finite topology.
- Each annular end of $M$ is asymptotic to a totally umbilic immersed annulus of finite area.
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Theorem (Adams-Meeks-Ramos(2018))

- Let $H \geq 0$ and $M$ be a connected noncompact surface of finite topology and negative Euler characteristic.
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- Let $H \geq 0$ and $M$ be a connected noncompact surface of finite topology and negative Euler characteristic.
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  - There does NOT exist a complete hyperbolic 3-manifold of finite volume containing a proper embedding of $M$ with constant mean curvature $H \geq 1$.
Theorem (The Chain Lemma)

Let $L$ be a link in a 3-manifold $M$ such that the link complement $M \setminus L$ admits a complete hyperbolic metric of finite volume. Suppose that there is a sphere $S$ in $M$ bounding a ball $B$ that intersects $L$ as in Figure 2 (a). Let $L'$ be the resulting link obtained by replacing $L \cap B$ by the components as appear in Figure 2 (b). Then $M \setminus L'$ admits a complete hyperbolic metric of finite volume.

Figure: Replacing (a) with (b) preserves hyperbolicity of the complement.
Theorem (The Switch Move Lemma)

Let $L$ be a link in a 3-manifold $M$ such that $M \setminus L$ admits a complete hyperbolic metric of finite volume. Let $\alpha \subset M$ be the closure in $M$ of a complete, properly embedded geodesic of $M \setminus L$ with distinct endpoints on $L$. Let $B$ be a closed ball in $M$ containing $\alpha$ in its interior and such that $B \cap L$ is composed of two arcs in $L$, as in Figure 3. Let $L_1$ be the resulting link in $M$ obtained by replacing $L \cap B$ by the components as appearing in Figure 4 (b). Then $M \setminus L_1$ admits a complete hyperbolic metric of finite volume.
Figure: The trace of a geodesic $\alpha$ of $(M \setminus L, h)$ joins distinct components $G, G'$ of $L$, and a neighborhood $B$ of $\alpha$ intersects $L$ in two arcs $g \subset G$ and $g' \subset G'$.