Arnold diffusion for ‘complete’ families of perturbations with two or three independent harmonics

Emerging interactions of geometric and variational methods

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Consider a pendulum and a rotor plus a time periodic perturbation depending on two harmonics in the variables \((\varphi, s)\):

\[
H_\varepsilon(p, q, I, \varphi, s) = \pm \left( \frac{p^2}{2} + \cos q - 1 \right) + \frac{l^2}{2} + \varepsilon h(q, \varphi, s) \quad (1)
\]

\[
h(q, \varphi, s) = f(q)g(\varphi, s), \quad f(q) = \cos q,
\]

\[
g(\varphi, s) = a_1 \cos(k_1 \varphi + l_1 s) + a_2 \cos(k_2 \varphi + l_2 s), \quad (2)
\]

for some \(k_1, k_2, l_1, l_2 \in \mathbb{Z}\).

**Theorem**

Assume that \(a_1 a_2 \neq 0\) and \(\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0\) in (1)-(2). Then, for any \(I^* > 0\), there exists \(\varepsilon^* = \varepsilon^*(I^*, a_1, a_2) > 0\) such that for any \(\varepsilon, 0 < \varepsilon < \varepsilon^*\), there exists a trajectory \((p(t), q(t), I(t), \varphi(t))\) such that for some \(T > 0\)

\[
I(0) \leq -I^* < I^* \leq I(T).
\]

**Remark:** \(I(t) \equiv \text{constant for } \varepsilon = 0\).
The a priori unstable system

Goals

- To review the construction of scattering maps initiated in [D-Llave-Seara00], designed to detect global instability.
- To compute explicitly several scattering maps to prove global instability for the action \( I \) for any \( \varepsilon > 0 \) small enough.
- To estimate the time of diffusion in some cases (at least for \( k_1 = l_2 = 1 \) and \( l_1 = k_2 = 0 \)).
- To play with the parameter \( \mu = a_1/a_2 \) to prove global instability for any value of \( \mu \neq 0, \infty \).
- To describe bifurcations of the scattering maps.
- To get a glimpse of the \( 3 + \frac{1}{2} \) degrees of freedom case.
It is easy to check that if

$$\Delta := k_1 l_2 - k_2 l_1 = 0 \quad \text{or} \quad a_1 = 0 \quad \text{or} \quad a_2 = 0$$

there is no global instability for the variable $l$.

If $\Delta a_1 a_2 \neq 0$, after some rational linear changes in the angles, we only need to study two cases:

- The first (and easier) case [D-Schaefer17]

$$g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$$

- The second case [D-Schaefer17a]

$$g(\varphi, \sigma) = a_1 \cos \varphi + a_2 \cos \sigma,$$

where $\sigma = \varphi - s$. 
We deal with an a priori unstable Hamiltonian [Chierchia-Gallavotti94].

In the unperturbed case $\varepsilon = 0$, the Hamiltonian $H_0$ is integrable formed by the standard pendulum plus a rotor

$$H_0(p, q, l, \varphi, s) = \pm \left( \frac{p^2}{2} + \cos q - 1 \right) + \frac{l^2}{2}.$$ 

$I$ is constant: $\triangle l := l(T) - l(0) \equiv 0$.

For any $0 < \varepsilon \ll 1$, there is a finite drift in the action of the rotor $l$: $\triangle l = \mathcal{O}(1)$, so we have global instability.

In short, this is is also frequently called Arnold diffusion.
Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct iterates under several **Scattering maps** and the **Inner map**, giving rise to diffusing **pseudo-orbits**.
- To use previous results about Shadowing [Fontich-Martín00], [Gidea-Llave-Seara14]) for ensuring the existence of real orbits close to the pseudo-orbits.
We have two important dynamics associated to the system: the inner and the outer dynamics on a large invariant object $\tilde{\Lambda}$:

$$\tilde{\Lambda} = \{(0, 0, l, \varphi, s); l \in [-l^*, l^*], (\varphi, s) \in \mathbb{T}^2\},$$

which is a 3D *Normally Hyperbolic Invariant Manifold* (NHIM) with associated 4D stable $W^s_\varepsilon(\tilde{\Lambda})$ and unstable $W^u_\varepsilon(\tilde{\Lambda})$ invariant manifolds.

- The **inner dynamics** is the dynamics restricted to $\tilde{\Lambda}$. (Inner map)
- The **outer dynamics** is the dynamics along the invariant manifolds to $\tilde{\Lambda}$. (Scattering map)

Remark: Due to the form of the perturbation, $\tilde{\Lambda} = \tilde{\Lambda}_\varepsilon$. 
For the first case $g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$, the inner dynamics is described by the Hamiltonian systems with the Hamiltonian

$$K(I, \varphi, s) = \frac{I^2}{2} + \varepsilon (a_1 \cos \varphi + a_2 \cos s).$$

In this case the inner dynamics is integrable (a pendulum).
For $g(\varphi, \sigma)$, the inner dynamics is by the Hamiltonian

$$K(I, \varphi, \sigma) = \frac{I^2}{2} + \varepsilon (a_1 \cos \varphi + a_2 \cos \sigma),$$

where $\sigma = \varphi - s$. The system associated to this Hamiltonian is not integrable and two resonances arise in $I = 0$ and $I = 1$. 
Let \( \tilde{\Lambda} \) be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold \( \Gamma \). A scattering map is a map \( S \) defined by 
\[
S(\tilde{x}_-) = \tilde{x}_+ \text{ if there exists } \tilde{z} \in \Gamma \text{ satisfying}
\]
\[
|\phi^\varepsilon_t(\tilde{z}) - \phi^\varepsilon_t(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \pm\infty
\]
that is, \( W^u_\varepsilon(\tilde{x}_-) \) intersects transversally \( W^s_\varepsilon(\tilde{x}_+) \) in \( \tilde{z} \).
Outer dynamics

Scattering map

\( S \) is symplectic and exact [D-Llave-Seara08] and takes the form:

\[
S_\varepsilon(I, \varphi, s) = \left( I + \varepsilon \frac{\partial L^*}{\partial \theta}(I, \theta) + O(\varepsilon^2), \theta - \varepsilon \frac{\partial L^*}{\partial I}(I, \theta) + O(\varepsilon^2), s \right),
\]

where \( \theta = \varphi - Is \) and \( L^*(I, \theta) \) is the Reduced Poincaré function, or more simply in the variables \((I, \theta)\):

\[
S_\varepsilon(I, \theta) = \left( I + \varepsilon \frac{\partial L^*}{\partial \theta}(I, \theta) + O(\varepsilon^2), \theta - \varepsilon \frac{\partial L^*}{\partial I}(I, \theta) + O(\varepsilon^2) \right),
\]

- The variable \( s \) remains fixed under \( S_\varepsilon \): it plays the role of a parameter
- Up to first order in \( \varepsilon \), \( S_\varepsilon \) is the \( -\varepsilon \)-time flow of the Hamiltonian \( L^*(I, \theta) \)
- The scattering map jumps \( O(\varepsilon) \) distances along the level curves of \( L^*(I, \theta) \)
To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

**Proposition**

Given $(l, \varphi, s) \in [-l^*, l^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \mapsto \mathcal{L}(l, \varphi - l \tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(l, \varphi, s)$, where

$$\mathcal{L}(l, \varphi, s) = \int_{-\infty}^{+\infty} (\cos q_0(\sigma) - \cos 0) g(\varphi + l\sigma, s + \sigma; 0) d\sigma.$$  

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\tilde{\Lambda}_\varepsilon$, which is $\varepsilon$-close to the point $\tilde{z}^*(l, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), l, \varphi, s) \in \mathcal{W}^0(\tilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(l, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), l, \varphi, s) \in \mathcal{W}^u(\tilde{\Lambda}_\varepsilon) \cap \mathcal{W}^s(\tilde{\Lambda}_\varepsilon).$$
In our model \( q_0(t) = 4 \arctan e^t, p_0(t) = 2 / \cosh t \) is the separatrix for positive \( p \) of the standard pendulum \( P(q, p) = p^2 / 2 + \cos q - 1 \).

- For \( g(\phi, s) = a_1 \cos \phi + a_2 \cos s \), the Melnikov potential becomes
  \[
  \mathcal{L}(I, \phi, s) = A_1(I) \cos \phi + A_2 \cos s,
  \]
  where \( A_1(I) = \frac{2 \pi I a_1}{\sinh \left( \frac{I \pi}{2} \right)} \) and \( A_2 = \frac{2 \pi a_2}{\sinh \left( \frac{\pi}{2} \right)} \).

- For \( g(\phi, \sigma) = a_1 \cos \phi + a_2 \cos \sigma \ (\sigma = \phi - s) \), the Melnikov potential becomes
  \[
  \mathcal{L}(I, \phi, \sigma) = A_1(I) \cos \phi + A_2(I) \cos \sigma,
  \]
  where \( A_1(I) \) is as before but now \( A_2(I) = \frac{2 \left( I - 1 \right) \pi a_2}{\sinh \left( \frac{\left( I - 1 \right) \pi}{2} \right)} \).
The Melnikov potentials are similar in both cases.

Figure: The Melnikov Potential, $\mu = a_1/a_2 = 0.6$, $l = 1$, $g(\varphi, s)$. 
Finally, the function $\mathcal{L}^*(l, \theta)$ can be defined:

**Definition**

The Reduced Poincaré function is

$$\mathcal{L}^*(l, \theta) = \mathcal{L}(l, \varphi - l \tau^*(l, \varphi, s), s - \tau^*(l, \varphi, s)),$$

where $\theta = \varphi - l s$.

Therefore the definition of $\mathcal{L}^*(l, \theta)$ depends on the function $\tau^*(l, \varphi, s)$. 
From the Proposition given above, we look for $\tau^*$ such that
\[
\frac{\partial \mathcal{L}}{\partial \tau}(l, \varphi - l \tau^*, s - \tau^*) = 0.
\]

**Different view-points for $\tau^* = \tau^*(l, \varphi, s)$**

- Look for critical points of $\mathcal{L}$ on the straight line, called NHIM line $R(l, \varphi, s) = \{(\varphi - l \tau, s - \tau), \tau \in \mathbb{R}\}$.
- Look for intersections between $R(l, \varphi, s) = \{(\varphi - l \tau, s - \tau), \tau \in \mathbb{R}\}$ and a crest which is a curve of equation
\[
\frac{\partial \mathcal{L}}{\partial \tau}(l, \varphi - l \tau, s - \tau)|_{\tau=0} = 0.
\]

Note that the crests are characterized by $\tau^*(l, \varphi, s) = 0$. 
Definition - Crests [D-Huguet11]

For each \( I \), we call crest \( C(I) \) the set of curves in the variables \((\varphi, s)\) of equation

\[
I \frac{\partial L}{\partial \varphi}(I, \varphi, s) + \frac{\partial L}{\partial s}(I, \varphi, s) = 0.
\]

which in our case can be rewritten as

\[
g(\varphi, s): \mu \alpha(I) \sin \varphi + \sin s = 0, \quad \text{with } \alpha(I) = \frac{I^2 \sinh(\frac{\pi I}{2})}{\sinh(\frac{\pi}{2})}, \quad \mu = \frac{a_1}{a_2}.
\]

\[
g(\varphi, \sigma = \varphi - s): \mu \alpha(I) \sin \varphi + \sin \sigma = 0, \quad \text{with } \alpha(I) = \frac{I^2 \sinh(\frac{(I-1)\pi I}{2})}{(I-1)^2 \sinh(\frac{\pi I}{2})}, \quad \mu = \frac{a_1}{a_2}.
\]

- For any \( I \), the critical points of the Melnikov potential \( L(I, \cdot, \cdot) \) ((0, 0), (0, \( \pi \)), \( (\pi, 0) \) and \( (\pi, \pi) \): one maximum, one minimum point and two saddle points) always belong to the crest \( C(I) \).
- \( L^*(I, \theta) \) is nothing else but \( L \) evaluated on the crest \( C(I) \).
- \( \theta = \varphi - Is \) is constant on the NHIM line \( R(I, \varphi, s) \)
Figure: Level curves of $\mathcal{L}$ for $\mu = a_1/a_2 = 0.5$, $l = 1.2$ and $g(\varphi, s)$. 
Understanding the behavior of the crests

\[ \downarrow \]

Understanding the behavior of the Reduced Poincaré function

\[ \downarrow \]

Understanding the Scattering map
For $|\mu_\alpha(I)| < 1$, there are two crests $C_{M,m}(I)$ parameterized by:

\[
\begin{align*}
    s &= \xi_M(I, \varphi) = -\arcsin(\mu_\alpha(I) \sin \varphi) \mod 2\pi \\
    \xi_m(I, \varphi) &= \arcsin(\mu_\alpha(I) \sin \varphi) + \pi \mod 2\pi
\end{align*}
\]

They are "horizontal" crests.
First case: \( g(\varphi, s) \) \( 0 < |\mu| < 0.625 \)

- For each \( I \), the NHIM line \( R(I, \varphi, s) \) and the crest \( C_{M,m}(I) \) has only one intersection point.

- The scattering map \( S_M \) associated to the intersections between \( C_{M}(I) \) and \( R(I, \varphi, s) \) is well defined for any \( \varphi \in \mathbb{T} \). Analogously for \( S_m \), changing \( M \) to \( m \). In the variables \( (I, \theta = \varphi - Is) \), both scattering maps \( S_M, S_m \) are globally well defined.

(a) Level curves of \( \mathcal{L}^*_M(I, \theta) \)  
(b) Level curves of \( \mathcal{L}^*_m(I, \theta) \)
First case: \( g(\varphi, s) \)

- There are **tangencies** between \( C_{M,m}(I, \varphi) \) and \( R(I, \varphi, s) \). For some value of \((I, \varphi, s)\), there are 3 points in \( R(I, \varphi, s) \cap C_{M,m}(I) \).

- This implies that there are 3 scattering maps associated to each crest with different domains. (**Multiple Scattering maps**)
First case: $g(\varphi, s)$  

$0.625 < |\mu|$  

(c) The three types of level curves.  
(d) Zoom where the scattering maps are different  

**Figure:** Level curves of $\mathcal{L}_M^*(l, \theta)$, $\mathcal{L}_M^{* (1)}(l, \theta)$ and $\mathcal{L}_M^{* (2)}(l, \theta)$
For some values of $I$, $|\mu \alpha(I)| > 1$, the two crests $C_{M,m}$ are parameterized by:

\[
\begin{align*}
\varphi &= \eta_M(I, s) = -\arcsin(\mu \alpha(I) \sin s) \pmod{2\pi} \\
\eta_m(I, s) &= \arcsin(\mu \alpha(I) \sin s) + \pi \pmod{2\pi}
\end{align*}
\] (5)

They are “vertical” crests.
First case: \( g(\varphi, s) \), \(|\mu| > 0.97\)

For the values of \( l \) for which horizontal crests become vertical, it is not always possible to prolong in a continuous way the scattering maps, so the domain of the scattering map has to be restricted.

**Figure:** The level curves of \( \mathcal{L}_M^*(l, \theta), \mu = 1.5 \).

In green, the region where the scattering map \( S_M \) is not defined.
Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I, \theta) = A_2 = \frac{2\pi a_2}{\sinh(\pi/2)}.$$

- The highways are “vertical” in the variables $(\varphi, s)$
- We always have a pair of highways. One goes up, the other goes down (this depends on the sign of $\mu = a_1/a_2$)
- The highways give rise to fast diffusing pseudo-orbits
First case: \( g(\varphi, s) \)

Figure: The scattering map jumps \( O(\varepsilon) \) distances along the level curves of \( L^*(l, \theta) \)
First case: $g(\varphi, s)$

An example of pseudo-orbit

Figure: In red: Inner map, blue: Scattering map, black: Highways
An estimate of the total time of diffusion between $-I^*$ and $I^*$, along the highway, is

$$T_d = \frac{T_s}{\varepsilon} \left[ 2 \log \left( \frac{C}{\varepsilon} \right) + O(\varepsilon^b) \right], \text{ for } \varepsilon \to 0, \text{ where } 0 < b < 1,$$

with

$$T_s = T_s(I^*, a_1, a_2) = \int_0^{I^*} \frac{-\sinh(\pi I/2)}{\pi a_1 I \sin \psi_h(I)} dI,$$

where $\psi_h = \theta - I \tau^*(I, \theta)$ is the parameterization of the highway $\mathcal{L}^*(I, \psi_h) = A_2$, and

$$C = C(I^*, a_1, a_2) = 16 |a_1| \left( 1 + \frac{1.465}{\sqrt{1 - \mu^2 A^2}} \right)$$

where $A = \max_{I \in [0, I^*]} \alpha(I)$, with $\alpha(I) = \frac{\sinh(\pi I/2)}{\sinh(\pi I/2)}$ and $\mu = a_1/a_2$.

**Note:** This estimate agrees with the upper bounds given in [Bessi-Chierchia-Valdinoci01] and quantifies the general optimal diffusion estimate $O \left( \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \right)$ of [Berti-Biasco-Bolle03] and [Treschev04].
Second case: $g(\varphi, \sigma), \sigma = \varphi - s$

Main differences

In the second case:

- For $|\mu \alpha(I)| < 1$, there are two crests $C_{M,m}(I)$ parameterized by $\sigma = \xi_M(I, \varphi)$ and $\xi_m(I, \varphi)$. For $|\mu \alpha(I)| > 1$, $C_{M,m}(I)$ parameterized by $\varphi = \eta_M(I, \sigma)$ and $\eta_m(I, \sigma)$. The crests lie on the plane $(\varphi, \sigma)$.

- There are no global Highways.

- For any value of $\mu = a_1/a_2$ is possible to find $I_h$ and $I_v$ such that for $I = I_h$ the crests are horizontal and for $I = I_v$ the crests are vertical.

- For any value of $\mu$ there exists $I$ such that the crests and some NHIM line are tangent. There are always multiple scattering maps.
Second case: \( g(\varphi, \sigma), \sigma = \varphi - s \)

From the definitions of \( R(I, \varphi, s) \) and \( C(I) \), we have

\[
R(I, \varphi, s) \cap C(I) = \{(I, \varphi - l\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s))\}.
\]

Introducing

\[
\tau^*(I, \theta) := \tau^*(I, \varphi - Is, 0), \quad \text{with} \quad \theta = \varphi - Is = (1 - l)\varphi + l\sigma,
\]

one can see that on the plane \((\varphi, \sigma = \varphi - s)\), the NHIM lines take the form

\[
R_I(\varphi, \sigma) = \{(\varphi - l\tau, \sigma - (1 - 1)\tau), \tau \in \mathbb{R}\}
\]

and that

\[
R_I(\varphi, \sigma) \cap C(I) = \{(\theta - l\tau^*(I, \theta), \theta - (1 - 1)\tau^*(I, \theta))\}.
\]

Therefore, the function \( \tau^*(I, \theta) \) is the time spent to go from a point \((\theta, \theta)\) in the diagonal \( \sigma = \varphi \) up to \( C(I) \) with a velocity vector \( \mathbf{v} = -(l, l - 1) \).
Second case: \( g(\varphi, \sigma), \sigma = \varphi - s \)

The choice of the concrete curve of the crest and therefore of \( \tau^*(I, \theta) \) is very important and useful.

**Figure:** Going down along NHIM lines

- **Green** zones: \( I \) increases under the scattering map.
- **Red** zones: \( I \) decreases under the scattering map.

**Figure:** The “lower” crest
Second case: \( g(\varphi, \sigma), \sigma = \varphi - s \)

Kinds of scattering maps

**Figure:** Going up along NHIM lines

**Figure:** The “upper” crest
Second case: $g(\varphi, \sigma), \sigma = \varphi - s$

Kinds of scattering maps

Figure: Minimal time

Figure: Minimal $|\tau^*|$ between “lower” and “upper” crest
Second case: \( g(\varphi, \sigma), \sigma = \varphi - s \)

Piecewise smooth \( S(I, \theta) \)

In this picture we show a combination of 3 scattering maps.

**Figure:** First intersection

**Figure:** Minimal \( |\tau^*| \) between \( C_M(I) \) and \( C_m(I) \)
Consider a pendulum and two rotors plus a time periodic perturbation depending on three harmonics in the angles \((\varphi_1, \varphi_2, \varphi_3 = s)\):

\[
H_\varepsilon(p, q, I_1, I_2, \varphi_1, \varphi_2, s) = \pm \left( \frac{p^2}{2} + \cos q - 1 \right) + h(I_1, I_2) + \varepsilon f(q) g(\varphi_1, \varphi_2, s),
\]

\[
h(I_1, I_2) = \Omega_1 I_1^2/2 + \Omega_2 I_2^2/2, \quad f(q) = \cos q
\]

\[
g(\varphi_1, \varphi_2, s) = a_1 \cos \varphi_1 + a_2 \cos \varphi_2 + a_3 \cos s.
\]

**Theorem (Arnold diffusion for a two-parameter family)**

Assume \(a_1 a_2 a_3 \neq 0\) and \(|a_1/a_3| + |a_2/a_3| < 0.625\) in Hamiltonian (6)+(7). Then, for any two actions \(I_\pm\) and any \(\delta\) there exists \(\varepsilon_0 > 0\) such that for every \(0 < |\varepsilon| < \varepsilon_0\) there exists an orbit \(\tilde{x}(t)\) and \(T > 0\) such that

\[
|I(\tilde{x}(0)) - I_-| \leq \delta \quad \text{and} \quad |I(\tilde{x}(T)) - I_+| \leq \delta
\]
For $|a_1/a_3| + |a_2/a_3| < 0.625$ there are two horizontal crests $C_{M,m}(I)$, and both scattering maps $S_M, S_m$ are globally well defined.

**Figure:** Horizontal crests: $a_1/a_3 = a_2/a_3 = 0.48, \Omega_1 l_1 = \Omega_2 l_2 = 1.219$.

Diffusing orbits are found by shadowing orbits of both scattering maps scattering maps and the inner dynamics.

**Remark**

*Actually, we can prove that given any two actions $I_{\pm}$ and any path $\gamma(s)$ joining them in the actions space, there exists an orbit $\tilde{x}(t)$ such that $I(\tilde{x}(t))$ is $\delta$-close to $\gamma(\Psi(t))$ for some parameterization $\Psi$.***
Theorem (Diffusion paths using only Scattering maps)

Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$ in Hamiltonian (6)+(7).

Given any two $(I_+, \theta_+) \in \tilde{\mathcal{I}}$, where

$$\tilde{\mathcal{I}} = \mathbb{R}^2 \times \mathbb{T}^2 \setminus \{(0, 0, 0, 0), (0, 0, \pi, 0), (0, 0, 0, \pi), (0, 0, \pi, \pi)\},$$

and any $\delta$ there exists $\varepsilon_0 > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$ there is an orbit $(I^i, \theta^i)_{0 \leq i < N}$ of the polyscattering map $(S_0, S_1, S_2)$:

$$(I^{i+1}, \theta^{i+1}) = S_\ell(I^i, \theta^i), \text{ where } \ell \in \{0, 1, 2\},$$

such that

$$|(I^0, \theta^0) - (I_-, \theta_-)| < \delta \text{ and } |(I^N, \theta^N) - (I_+, \theta_+)| < \delta.$$
Theorem (Existence of Highways)

Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$ in Hamiltonian (6)+(7). Given any $0 < c_j < C_j$, $j = 1, 2$, there is an orbit $(l^i, \theta^i)_{0 \leq i < N}$ of the scattering map $S_0$ such that

$$|l_j^0| < c_j \quad \text{and} \quad |l_j^N| > C_j, \quad j = 1, 2.$$