Edge behavior of deformed Wigner matrices

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Wigner matrix

Definition
A Hermitian Wigner matrix of size $N$ is a Hermitian random matrix

$$W = (w_{ij}) = \frac{1}{\sqrt{N}}(x_{ij}), \quad (1 \leq i, j \leq N),$$

whose entries $(x_{ij})$ are complex random variables, independent up to the constraint $x_{ij} = \overline{x}_{ji}$, such that $\mathbb{E} w_{ij} = 0$, and

$$\mathbb{E}|w_{ij}|^2 = \frac{1}{N}, \quad \mathbb{E} w_{ij}^2 = \begin{cases} \frac{1}{N}, & i = j, \\ 0, & i \neq j. \end{cases}$$

For simplicity, assume subexponential decay of the matrix entries,

$$\mathbb{P} (|x_{ij}| > s) \leq C_0 e^{-s^{\vartheta}},$$

for some constants $C_0$ and $\vartheta > 0$.

Special case: Gaussian unitary ensemble, c.f., Bourgade’s talk.

Then the eigenvalues of $W$ follow the semicircle law, Wigner [1955]:
Wigner’s Semicircle law

Histogram of the eigenvalues of a $N = 5000$ Hermitian Wigner matrix with complex Gaussian entries
Deformed Wigner matrix

- Let $V = \text{diag} (v_i)$ be an $N \times N$ diagonal random matrix whose entries are real, centered, i.i.d. random variables.
- Assume that the distribution of $(v_i)$ is
  \[
  \mu(v) = Z^{-1} (1 + v)^b (1 - v)^b f(v) \chi_{[-1,1]}(v),
  \]
  where
  \[-1 < b < \infty, \quad f \in C^1, \quad \text{with } f(v) > 0, \quad Z \text{ is a normalization}.
  \]
  In particular: $\mathbb{E} v_i = 0$ and $\mathbb{E} v_i^2 = O(1)$.

Deformed Wigner matrix / Wigner matrix with random potential

For $\lambda \in \mathbb{R}^+$, set
  \[
  H = (h_{ij}) := \lambda V + W,
  \]
and assume that $V$ and $W$ are independent.

- Note $\lambda = O(N^0)$, so that the eigenvalues of $\lambda V$ and $W$ are of the same order.
- Then the eigenvalues follow the deformed semicircle law, $\mu_{fc}$, Pastur [1972]:
Example 1: \((v_i)\) are distributed according to

\[
\mu(v) = Z^{-1}(1 - v)^{\frac{1}{3}}(1 + v)^{\frac{1}{3}} \chi[-1,1](v):
\]

Histogram of eigenvalues of a \(N = 5000\) deformed Wigner matrix with \(\lambda = 2\), respectively \(\lambda = 4\).
Stieltjes transform & deformed semicircle law I

- Stieltjes transform of a measure $\omega$,

$$m_\omega(z) := \int_{\mathbb{R}} \frac{d\omega(x)}{x - z}, \quad z = E + i\eta \in \mathbb{C}^+.$$ 

- Inversion formula (for abs. continuous $\omega$)

$$\omega(E) = \lim_{\eta \searrow 0} \frac{1}{\pi} \text{Im} \ m_\omega(E + i\eta).$$

For example:

$$d\mu_{sc}(x) := \frac{1}{2\pi} \sqrt{4 - x^2} \chi_{[-2,2]}(x) dx,$$

$$m_{\mu_{sc}}(z) = -\frac{1}{z + m_{\mu_{sc}}(z)}, \quad z \in \mathbb{C}^+,$$

with $\text{Im} \ m_{\mu_{sc}}(z) \geq 0$, $\eta > 0$. 
Stieltjes transform & deformed semicircle law II

\[ m_\omega(z) := \int_{\mathbb{R}} \frac{d\omega(x)}{x - z}, \quad m_{\mu_{sc}}(z) = -\frac{1}{z + m_{\mu_{sc}}(z)}, \quad z \in \mathbb{C}^+ . \]

- The Stieltjes transform of the deformed semicircle law, \( m_{\mu_{fc}} \), satisfies

\[ m_{\mu_{fc}}(z) = \int_{\mathbb{R}} \frac{d\mu(v)}{\lambda v - z - m_{\mu_{fc}}(z)} , \quad \text{(Pastur relation)} \]

and \( \text{Im} m_{\mu_{fc}}(z) \geq 0, \eta > 0. \)

- The deformed semicircle law, \( \mu_{fc} \), is then obtained through the inversion formula (\( \mu_{fc} \) is abs. continuous).

- Alternative definition, additive free convolution \( \mu_{fc} = \mu \boxplus \mu_{sc} \), c.f., free probability theory, Voiculescu,...[1985-...].

- For the special choice of \( \mu \) above, \( \mu_{fc} \) is supported on a single interval, \( \text{supp} \mu_{fc} = [L-, L+] \).
Eigenvector behavior I

Denote by \((\lambda_\alpha)\) the eigenvalues (with \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N\)), by \((u_\alpha)\) the associated \((\ell^2)\)-normalized eigenvectors and by \((u_\alpha(k))\) the components of the eigenvectors of \(H\). Then:

- \(\lambda = 0\) (Wigner matrix),

\[
|u_\alpha(k)| \lesssim N^{-1/2}, \quad (1 \leq \alpha \leq N, 1 \leq k \leq N).
\]

All eigenvectors are completely delocalized. 
Erdős-Schlein-Yau,...[2009-2012].

- \(\lambda \neq 0, \ b < 1\),

\[
|u_\alpha(k)| \lesssim N^{-1/2}, \quad (1 \leq \alpha \leq N, 1 \leq k \leq N),
\]

for all finite \(\lambda\). All eigenvectors are completely delocalized. 
Lee-S. [2013]
Deformed semicircle law II

Example 2: \((v_i)\) are distributed according to

\[
\mu(v) = Z^{-1} (1-v)^4 (1+v)^4 \chi_{[-1,1]}(v) : 
\]

Histograms of eigenvalues of a \(N = 5000\) deformed Wigner matrix with \(\lambda = 2\), respectively \(\lambda = 4\).
Eigenvector behavior II

For $b > 1$, there is a constant $\lambda_+$, such that we have

$$
\mu_{fc}(E) \sim \begin{cases} 
\sqrt{\kappa_E}, & \text{if } \lambda < \lambda_+, \\
(\kappa_E)^b, & \text{if } \lambda > \lambda_+, 
\end{cases} \quad E \geq 0,
$$

where $\kappa_E$ denotes the distance from $E$ to the upper endpoint of the support of $\mu_{fc}$. (A similar statement holds for $E \leq 0$).

Wlog assume that $v_1 \geq v_2 \geq \ldots \geq v_N$.

- For $\lambda < \lambda_+$, all eigenvectors are completely delocalized:
  $$
  |u_\alpha(k)|^2 \lesssim N^{-1}, \quad (1 \leq \alpha \leq N, 1 \leq k \leq N).
  $$

- For $\lambda > \lambda_+$, the eigenvectors in the bulk are completely delocalized; at the extreme edge they are ‘partially localized’, i.e.,
  $$
  |u_\alpha(\alpha)|^2 = \frac{\lambda^2 - \lambda_+^2}{\lambda^2} + o(1),
  $$
  $$
  |u_\alpha(k)|^2 \lesssim \frac{1}{N} \frac{1}{\lambda^2 |v_\alpha - v_k|^2}, \quad (\alpha \neq k, 1 \leq k \leq N),
  $$

where $\alpha \leq n_0$, for some fixed $n_0$. (Similar statement holds for the lower edge)

Lee-S.-Yau
Fluctuations of the largest eigenvalue

The largest eigenvalue $\lambda_1$ of $H$ approaches $L_+$, as $N \to \infty$, where $\text{supp} \mu_{fc} = [L_-, L_+]$.

Fluctuations at the (upper) edge: $\lambda = O(1)$

- In the delocalized regime,

  $$\lim_{N \to \infty} \mathbb{P} \left( N^{1/2} (L_+ - \lambda_1) \leq s \right) = \Phi_a(s), \quad s \in \mathbb{R},$$

  where $\Phi_a$ is the CDF of centered Gaussian of variance $a = a(\mu, \lambda)$;

- In the ‘partially localized’ regime,

  $$\lim_{N \to \infty} \mathbb{P} \left( N^{1/(b+1)} (L_+ - \lambda_1) \leq s \right) = G_{b+1}(s), \quad s \in \mathbb{R},$$

  where

  $$G_{b+1}(s) = \left( 1 - e^{-\left( \frac{s}{c} \right)^{b+1}} \right) \chi_{[0, \infty)}(s)$$

  is the CDF of a Weibull distribution with parameters $b + 1$ and $c = c(\mu, \lambda)$.

Lee-S.-Yau
From Tracy-Widom to Gaussian: $\lambda = o(N^0)$

- For $\lambda = 0$ (Wigner matrix),

\[
\lim_{N \to \infty} \mathbb{P} \left( N^{2/3} (\lambda_1 - 2) \leq s \right) = \exp \left( - \int_{-\infty}^{\infty} (x - s)q(x)^2 \, dx \right) =: F_2(s),
\]

where $q$ satisfies

\[
q'' = xq + 2q^3, \quad q(x) \sim \text{Ai}(x), \quad \text{as} \quad x \to \infty.
\]


- For $\lambda \neq 0$, in case $W$ is a GUE matrix, it is known that the Tracy-Widom law (1) holds true for $\lambda \ll N^{-1/6}$, Johansson [2007], T. Shcherbina [2011], and that the transition to Gaussian fluctuations occurs at $\lambda \sim N^{-1/6}$, Johansson [2007].