Hamiltonian Dynamics and Morse theory

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IAS
Classical Hamiltonian mechanics

- Hamiltonian formulation of classical mechanics describes mechanics in terms of position $q$ and momentum $p$
- The Hamiltonian

$$H(t, q, p) = \frac{1}{2m} \| p \|^2 + V(t, q)$$

describes total energy of system, i.e., kinetic plus potential energy.
- Equations of motion are given by

$$\frac{d}{dt} q = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{d}{dt} p = -\frac{\partial H}{\partial q}.$$ 

- Typical example: Kepler problem
  A planet under influence of gravity of the sun. Here, $V(q) = \frac{1}{\| q \|}$.
The symplectic setting

- Natural setting for Hamiltonian mechanics: a symplectic manifold.
- In classical mechanics, this is the cotangent bundle of the space of positions (fibers are the momentum coordinates).
- In general, we consider a manifold $M^{2n}$, equipped with a closed, non-degenerate 2-form $\omega$.
- Now we can formulate equations of motion:
  For a Hamiltonian $H: S^1 \times M \to \mathbb{R}$, define the vector field $X_H$ by
  \[ \omega(X_H, \cdot) = -dH. \]
  Then the motion is given by the flow equation
  \[ \dot{x}(t) = X_H(x(t)). \]
Examples

• Classical mechanics on $\mathbb{R}^{2n}$:
  • $\omega$ given by $\omega(v, w) = v^T J w$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
  • The equations of motion are now
    $$\dot{x}(t) = X_H(x(t)) = J \nabla H(x(t))$$
  • This reproduces the system of ODEs in classical mechanics

• Geodesic flow on a manifold $B$ is given by the Hamiltonian
  $$H(q, p) = \frac{1}{2} \|p\|^2$$ on $M = T^*B$,

  where $q$ position on $B$ and $p$ is the fiber coordinate.
Number of periodic orbits

• I am mainly interested in periodic orbits of the time-1-map.
• Fixed points
  (one-periodic orbits of the flow)

• Periodic points
  (periodic flow lines with integer period)

• Question: Does the symplectic manifold carry information about the number of periodic orbits?
• Answer: Yes, in many cases even infinitely many periodic orbits are known a priori for all Hamiltonian systems.
The Conley conjecture

**Theorem**

*On certain symplectic manifolds, every Hamiltonian $H$ has infinitely many periodic orbits.*

- known for closed manifolds with vanishing first Chern class, on cotangent bundles and $\mathbb{R}^{2n}$ for certain classes of Hamiltonians
- In particular, this theorem is true for classical systems with position $q$ on closed manifolds and momentum $p$ in fibers of the cotangent bundle, i.e. for Hamiltonians of the form $H(t, q, p) = \|p\|^2 + V(t, q)$.

Proofs on different classes of manifolds by Conley-Zehnder, Salamon-Zehnder, Franks-Handel, Hingston, Ginzburg, Ginzburg-Gürel, H., Gürel

- Counter example: rotation on $S^2$ has only two periodic points (the fixed points).
Morse theory

- Idea: Use gradient flow of a function
- Counting gradient flow lines defines a boundary operator and the resulting homology is the singular homology of the manifold.
- There is also a Morse-theoretic version of the cup product, which can be used to show existence of critical points.

Critical points generate chain complex

Gradient flow lines determine the boundary map
Critical points of smooth functions

**Definition**

The cuplength of a manifold $Z$ is defined by

$$\text{cuplength}(Z) := \max\{k \in \mathbb{N} \mid \alpha_1 \cup \ldots \cup \alpha_k \neq 0 \text{ for } \alpha_i \in H^{\geq 1}(Z)\}.$$ 

Then we have the following

**Theorem (Albers-H.)**

Fix a function $F$ with a non-degenerate critical submanifold $Z$ and a smooth function $h$ which is close to $F$. Then the function $h$ has at least $\text{cuplength}(Z) + 1$ critical points.

This method can also be applied to action functionals in Hamiltonian dynamics.
Applications in Hamiltonian dynamics
Floer homology

- Floer homology is Morse homology for action functional whose critical points are fixed points of Hamiltonian diffeomorphisms.
- The Morse theory proof for existence of critical points applies here and shows existence of fixed points for small Hamiltonians.
- By the same method, we can also show existence of Hamiltonian chords for certain submanifolds.

I hope to modify the Morse proof to make it work in other cases of symplectic geometry.
Contact geometry - Weinstein conjecture

- Contact geometry is in some sense the odd-dimensional version of symplectic geometry. Contact manifolds carry Reeb vector fields, whose flow has similar properties to the Hamiltonian flow.

**Conjecture**

**Weinstein conjecture:** Every closed contact manifold has at least one closed Reeb orbit.

- Known for contact 3-manifolds (Taubes)
- The standard contact structure $S^3$ has at least two periodic orbits. (Christofaro-Gardiner - Hutchings, Ginzburg - H. - Hryniewicz - Macarini, Lui - Long)
- Few things are known in higher dimensions → Possibly above method applies and give lower bound for number of Reeb orbits.
Thank you