On the Boltzmann equation without angular cut-off.

Robert Strain (University of Pennsylvania)
Collaborators:
Philip Gressman (University of Pennsylvania)
Vedran Sohinger (University of Pennsylvania)

Non-equilibrium Dynamics and Random Matrices Seminar,
IAS, School of Mathematics,
Tuesday, March 18, 2014
Overall theme: The full Boltzmann equation is a physical non-local geometric fractionally diffusive (somewhat degenerate) PDE.
The Boltzmann Equation, derived by Boltzmann in 1872 (and Maxwell 1866), models the behavior of a dilute gas.

- Gas is modeled as a large ($\gtrsim 10^{23}$) number of particles obeying the laws of Newtonian mechanics.
- Each particle has its own position and velocity. It travels in a straight line until it happens to collide with another particle.
- Only binary collisions are allowed.
- Collisions obey the physical conservation laws of momentum and energy.
- The Boltzmann equation is derived by taking an appropriate continuum (Boltzmann-Grad) limit.
- The Boltzmann equation is probabilistic, describing the evolution in time for an arbitrary particle in the ensemble with a given initial position and momentum.
Particles moving around in a box

Image/Movie Credit: wikipedia
The Boltzmann equation (1872)

The Boltzmann equation

\[ \partial_t F + v \cdot \nabla_x F = Q(F, F), \quad x \in \Omega, \; v \in \mathbb{R}^3, \; t \geq 0 \]

- \( F = F(t, x, v) \): probability density in (position, velocity)
- \( \Omega \subset \mathbb{R}^3 \): domain in space
- \( v \cdot \nabla_x F \): free transport term
- \( Q(F, F) \): collision operator, local in \((t, x)\), quadratic integral operator

Derived from **rarefied gas dynamics**: Maxwell 1860', Boltzmann 1872.
Physical conservation laws of energy and momentum

Given two particles, with velocities \( v, v_* \in \mathbb{R}^3 \), after a binary collision they have outgoing velocities \( v', v'_* \in \mathbb{R}^3 \).

- These obey the conservation of momentum and energy:

\[
\begin{align*}
  v + v_* &= v' + v'_*, \\
  |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2.
\end{align*}
\]

- Since there are six unknowns, \((v', v'_*)\), and four equations, the set of solutions can be parametrized on the sphere \( \sigma \in S^2 \):

\[
\begin{align*}
  v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\
  v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.
\end{align*}
\]

- Describes the aftermath of a binary collision probabilistically.
Boltzmann equation (1872) for \((x, v) \in \Omega \times \mathbb{R}^3\) is

\[
\frac{\partial F}{\partial t} + v \cdot \nabla_x F = Q(F, F)
\]

\(F(t, \cdot, \cdot)\) is the density function of particles in the phase space.

- The *Boltzmann collision operator* is local in \((t, x)\) as:

\[
Q(F, G)(v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma \ B(v - v_*, \sigma) \left[ F' G' - F G \right].
\]

Shorthand \(G = G(v), F_* = F(v_*), G' = G(v'), F'_* = F(v'_*)\).

- the pre-post collisional velocities \((v, v_*)\) and \((v', v'_*)\) satisfy

\[
\begin{align*}
v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\
v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.
\end{align*}
\]

The aftermath of a binary collision

A complete description of the aftermath of collisions requires more precise physics. There are two major physical approaches:

- **Hard spheres**: Here the particles are assumed to be literal billiard balls of vanishingly small diameter in the continuum limit. This assumption leads to collision physics in which there is most likely a large deviation in angle after collisions.

- **Inverse power-law interactions**: Here the particles are assumed to interact pairwise through inverse power law potentials, \( \phi(r) = r^{-p+1} \) for some \( p > 2 \). In this case, the preferred consequence of a collision is a glancing or grazing collision, meaning that the velocities change only a small amount.

- **The Coulomb potential** \( \phi(r) = r^{-1} \): this requires some additional limiting arguments and leads to what is called the Landau equation. It is a reasonably good physical model for the dynamics of plasma.
Boltzmann collision kernel: $B(\nu - \nu_*, \sigma)$

\[
Q(F, G)(\nu) = \int_{\mathbb{R}^3} dv_\ast \int_{S^2} d\sigma \ B(\nu - \nu_*, \sigma) \left[ F_\ast' G' - F_\ast G \right].
\]

J. C. Maxwell in 1866 computed $B(\nu - \nu_*, \sigma)$ from the potential:

\[
\phi(r) = r^{-(p-1)}, \quad p \in (2, +\infty).
\]

This kernel takes product form in its arguments as

\[
B(\nu - \nu_*, \sigma) = |\nu - \nu_*|^\gamma b(\cos \theta), \quad \cos \theta = \frac{\nu - \nu_*}{|\nu - \nu_*|} \cdot \sigma.
\]

The angular singularity in $\sigma$ is not locally integrable:

\[
\sin \theta b(\cos \theta) \approx \theta^{-1-2s}, \quad s = \frac{1}{p-1} \in (0, 1), \quad \forall \theta \in (0, \frac{\pi}{2}].
\]

- The kinetic factor $|\nu - \nu_*|^\gamma$ can be singular: $\gamma = \frac{p-5}{p-1} > -3$.
- Maxwell 1866: $|\langle Q(F, G), \phi \rangle_{L^2(\mathbb{R}^3)}| < \infty$. 

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On the Boltzmann equation without angular cut-off
\[ b(\cos \theta) \in L^\infty(S^2), \quad \text{or} \quad b(\cos \theta) \in L^1(S^2) \]

Vast majority of research on the Boltzmann equation from the 1960’s to say the year 2000 made these assumptions. These assumptions are still in widespread usage today.

- Cut-off versions of the Boltzmann equation are known to propagate singularities: see Boudin-Desvillettes (2000), Duan-Li-Yang (2008), Bernis-Desvillettes (2009).


- Full Boltzmann equation without cut-off (near Maxwellian) decays to equilibrium much faster - “diffusion improves the H-theorem” - Gressman-S, AMUXY
Old conjecture regarding the angular singularity

The Boltzmann collision operator operator without cut-off has been widely conjectured to “essentially behave” as

\[ F \mapsto Q(g, F) \sim (-\Delta_v)^s F + \text{lower order terms} \]

Linear intuition due to Carlo Cercignani in 1969

- Conjecture does not take collisional geometry into account. This is an isotropic “flat” Laplacian.
- This is correct to the order \( s \), however we can prove that

\[ F \mapsto Q(g, F) \sim \langle v \rangle^{\gamma+2s} (-\Delta_P)^s F + \text{lower order terms}. \]

- If \( \mathbb{R}^3 \) is identified with a paraboloid in \( \mathbb{R}^4 \) by means of the mapping \( v \mapsto (v, \frac{1}{2}|v|^2) \), then \( \Delta_P \) is the metric Laplacian on the paraboloid induced by the Euclidean metric on \( \mathbb{R}^4 \).

(Note that \( \langle v \rangle = \sqrt{1 + |v|^2} \))
Heat diffusion on the paraboloid
We introduce a weighted geometric fractional semi-norm: $\dot{N}^{s,\gamma}$

$$
|f|_{\dot{N}^{s,\gamma}}^2 \overset{\text{def}}{=} \int_{\mathbb{R}^3} \, dv \, \langle v \rangle^{\gamma + 2s + 1} \int_{d(v,v') \leq 1} \, dv' \, \frac{(f(v') - f(v))^2}{d(v,v')^{3+2s}}.
$$

- We use the metric on the “lifted” paraboloid

$$
d(v, v') = \sqrt{|v - v'|^2 + (|v|^2 - |v'|^2)^2/4},
$$

- The precise statement: $(|f|_{L^2_{\gamma}}^2 \overset{\text{def}}{=} \int_{\mathbb{R}^3} \, dv \, \langle v \rangle^{\gamma} |f(v)|^2.)$


$$
\langle Q(g, F), F \rangle_{L^2(\mathbb{R}^3_{\gamma})} \approx |F|_{\dot{N}^{s,\gamma}}^2 + |F|_{L^2_{\gamma}}^2 + \text{lower order terms}.
$$

We think of the function $g$ as a parameter; derivative lower bound above holds only assuming $g \in L \log L \cap L^1(B_R)$ locally.

(Recall $\langle v \rangle = \sqrt{1 + |v|^2}$)
Intuition from relativistic physics - “on a mass shell”

To “flatten” the diffusive effects of the collision operator we define $v' = (v'_1, v'_2, v'_3, v'_4) \in \mathbb{R}^4$, $v'_*, v_*$, with $|v'|^2 = (v'_1)^2 + (v'_2)^2 + (v'_3)^2$:

$$\int_{\mathbb{R}^3} d\nu' = \int_{\mathbb{R}^4} d\nu' \delta (v'_4 - |v'|^2/2)$$

We further consider the measure $(\nu = (v, |v|^2/2))$

$$\int d\mu = \int_{\mathbb{R}^4} d\nu' \int_{\mathbb{R}^4} d\nu_* \int_{\mathbb{R}^4} d\nu' \delta^{(4)} (\nu + \nu_* - v' - \nu'_*) \times \delta (v'_4 - |v'|^2/2) \delta ((\nu_*)_4 - |\nu_*|^2/2) \delta ((\nu'_*)_4 - |\nu'_*|^2/2)$$

Equivalent expression for the Boltzmann collision operator

$$Q(F, F)(\nu) = \int d\mu \ B(\nu - \nu_*, \sigma) \left[ F(\nu'_*)F(\nu') - F(\nu_*)F(\nu) \right].$$

Insight: Singularity not only $v' \to v$, but also $|v'|^2 \to |v|^2 (!!)$
Graphing the Chirp on the Parabola
Conservation Laws for the Boltzmann equation

- Conservation of mass, momentum and energy hold (formally):

\[
\int_{\Omega \times \mathbb{R}^3} \, dxdv \, \begin{pmatrix} 1 \\ \frac{v}{|v|^2} \end{pmatrix} \left( F(t, x, v) - F_0(x, v) \right) = 0
\]

- Equilibrium states are the Maxwellians:

\[
\mu(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{3/2}} \exp \left( -\frac{|v-u|^2}{2T} \right)
\]

Above the density = \( \rho \), the bulk velocity = \( u \), and the temperature = \( T \) are the fluid variables.

- The five conservation laws and the five parameter Maxwellian \( \implies \exists ! \) equilibrium state for any given initial condition
Boltzmann’s celebrated H-theorem: Entropy is increasing:

\[- \frac{d}{dt} \int_\Omega dx \int_{\mathbb{R}^3} dv \, F \log F \geq 0.\]

This is a manifestation of the second law of thermodynamics.

- Entropy is maximized by the Maxwellian equilibria:

\[\mu(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}.\]

\(\rho, u, T\): the density, bulk velocity and temperature of the gas

- Boltzmann himself predicted convergence in large time to these equilibria because of the H-theorem. The “proof” was however held back by “slight analytical difficulties”.

- Big open problem for mathematicians to prove!

- We have a rigorous proof in the close to equilibrium context.
Lots of previous results (this is a short incomplete list)

- Hilbert (approx 1900)
- ...
- Ukai (1974 near equilibrium for Hard spheres)
- Ellis-Pinsky (1975, fluid approximation...)
- Illner-Shinbrot (1980’s near Vacuum)
- Caflish (1980, in $\mathbb{T}_x^3$),
  Ukai-Asano (1982, in $\mathbb{R}_x^n$)
- DiPerna - Lions (1992)
- ...
- Mouhot-S (2006), Gressman-S, AMUXY, ...
- Duan-S (2011, 2012), Guo-Wang (2012), Sohinger-S ...
We consider the time evolution of perturbations:

\[ F(t, x, v) = \mu(v) + \sqrt{\mu(v)} f(t, x, v) \]

The global Maxwellian equilibrium is \( \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2} \).

- The perturbation \( f = f(t, x, v) \) evolves via the equation

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + Lf = \Gamma(f, f)
\]

\[
\Gamma(f, g) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} g), \quad Lf = -\Gamma(f, \sqrt{\mu}) - \Gamma(\sqrt{\mu}, f)
\]

- Boltzmann’s H-Theorem: \( F(t, x, v) \to \mu(v) \) as \( t \to \infty \), or equivalently \( f(t, x, v) \to 0 \) as \( t \to \infty \).

- The linear operator has a large null space: \( PL = LP = 0 \).

\[
Pf = af(t, x)\sqrt{\mu} + \sum_{i=1}^{3} b_i(t, x)v_i\sqrt{\mu} + cf(t, x)(|v|^2 - 3)\sqrt{\mu}.
\]
Definition of various Sobolev spaces

We first define a unified weight function

\[ w(v) \overset{\text{def}}{=} \begin{cases} \langle v \rangle, & \gamma + 2s \geq 0, \quad \text{“hard potentials”} \\ \langle v \rangle^{-\gamma - 2s}, & \gamma + 2s < 0, \quad \text{“soft potentials”} \end{cases} \]

Also spatial and velocity derivatives:

\[ \partial_\alpha^\beta = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3} . \]

For \( \ell, K \geq 0 \), we define the \( H^K_\ell(T^3 \times \mathbb{R}^3) \) space by

\[ \| h \|_{H^K_\ell(T^3 \times \mathbb{R}^3)}^2 \overset{\text{def}}{=} \sum_{|\alpha| + |\beta| \leq K} \| w^{\ell - |\beta|} \partial_\alpha^\beta h \|_{L^2(T^3 \times \mathbb{R}^3)}^2. \]

Weighted non-isotropic derivative space \( |h|_{N^{s,\gamma}_\ell}^2 \) is

\[ |w^{\ell} h|_{L^2_{\gamma + 2s}}^2 + \int_{\mathbb{R}^3} dv \langle v \rangle^{\gamma + 2s + 1} w^{2\ell}(v) \int_{\mathbb{R}^3} dv' \frac{(h' - h)^2}{d(v, v')^{3 + 2s}} 1_{d(v, v') \leq 1} \]

\( N^{s,\gamma}_\ell, K \): \( \| h \|_{N^{s,\gamma}_\ell, K(T^3 \times \mathbb{R}^3)}^2 \overset{\text{def}}{=} \sum_{|\alpha| + |\beta| \leq K} \| \partial_\alpha^\beta h \|_{N^{s,\gamma}_{\ell - |\beta|}(T^3 \times \mathbb{R}^3)}^2. \)

\( L^2(\mathbb{R}^3) \)-norms in the velocity variables are denoted \( | \cdot |_{L^2} \).
Global classical solutions of the Boltzmann equation

Theorem (Gressman-S, J. Amer. Math. Soc., 2011)

Fix $K \geq 4$ the number of derivatives, and $\ell \geq 0$ the velocity weight. Choose $f_0(x, v) \in H^K_\ell (\mathbb{T}^3 \times \mathbb{R}^3)$ which satisfies the conservation laws. $\exists \eta_0 > 0$ such that if $\| f_0 \|_{H^K_\ell} \leq \eta_0$, then there exists a unique global classical solution to the Boltzmann equation which satisfies

$$f(t, x, v) \in L^\infty_t H^K_\ell \cap L^2_t N^{s, \gamma}_{\ell, K}((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3).$$

When $\gamma + 2s \geq 0$, for a fixed $\lambda > 0$, we have exponential decay

$$\| f(t) \|_{H^K_\ell (\mathbb{T}^3 \times \mathbb{R}^3)} \lesssim e^{-\lambda t} \| f_0 \|_{H^K_\ell (\mathbb{T}^3 \times \mathbb{R}^3)}.$$

When $\gamma + 2s < 0$, for any $m > 0$ we have polynomial decay

$$\| f(t) \|_{H^K_\ell (\mathbb{T}^3 \times \mathbb{R}^3)} \leq C_m (1 + t)^{-m} \| f_0 \|_{H^K_{\ell+m} (\mathbb{T}^3 \times \mathbb{R}^3)}.$$

Also positivity, i.e. $F = \mu + \sqrt{\mu} f \geq 0$ if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$. 

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Other global results without cut-off

Alexandre-Villani (CPAM, 2002) - large data renormalized weak solutions (ala Diperna-Lions) with defect measure $\mu \geq 0$:

$$\frac{\partial \beta(F)}{\partial t} + \mathbf{v} \cdot \nabla_x \beta(F) = \beta'(F) Q(F,F) + \mu.$$ 

Typically renormalize as for instance: $\beta(F) = \frac{F}{1+F}$.

Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY) (several publications 2011-2012) - global existence for small data in the whole space

- Very different methods. Their norm is, $|||g|||^2$, given by

$$\int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} d\sigma \ B(v-v_*, \sigma) \left\{ \mu_* (g' - g)^2 + g^2 \left( \sqrt{\mu'} - \sqrt{\mu} \right)^2 \right\}.$$ 

- $K \geq 6$ derivatives.
Our proof is built on the foundation laid by Guo and S for the Landau equation (corresponding to a limiting case $p \to 2$):

- Asymptotic time decay rates due to S-Guo (CPDE, 2006), (ARMA, 2008)

The Landau machinery breaks down at the following stage:

**Missing piece:** Find a Hilbert space $X$ (only in $\mathbb{R}^3_v$) such that

$$|\langle \Gamma(g, h), f \rangle| \lesssim |g|_{L^2} |h|_X |f|_X + \text{l.o.t.}$$

$$\langle Lf, f \rangle \gtrsim |f|_X^2 - \text{l.o.t.}$$

If such a Hilbert space $X$ exists, it is essentially unique.
General very high level strategy of proof

\[
\frac{\partial f}{\partial t} + \nu \cdot \nabla_x f + Lf = \Gamma(f, f).
\]

Consider the high order norm

\[
G(f)(t) \overset{\text{def}}{=} \|f(t)\|_{H^K(\mathbb{T}^3 \times \mathbb{R}^3)}^2 + \int_0^t \sum_{|\alpha| \leq K} \|\partial^\alpha f(\tau)\|_{N^s, \gamma}^2 d\tau.
\]

We have the energy inequality

\[
\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}^2 + (f, Lf)_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} = (f, \Gamma(f, f))_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}
\]

Key steps to prove non-linear stability: \( G(f)(t) \lesssim \|f_0\|_{H^K(\mathbb{T}^3 \times \mathbb{R}^3)}^2 \).

- Coercive estimate: \((f, Lf)_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \gtrsim \|f\|_{N^s, \gamma}^2 \). This is false as an operator inequality, only true for solutions.
- Non-linear estimate: \(|(f, \Gamma(f, f))_{L^2}| \lesssim \|f\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \|f\|_{N^s, \gamma}^2 \).
Split the linearized collision operator $L = N + K$, where $N$ is essentially the “norm” and $K$ is a “compact perturbation”:

**Lemma (Norm estimate)**

\[
\langle Ng, g \rangle \approx |g|^{2}_{N^{s, \gamma}}.
\]

**Lemma (Compact estimates, no derivatives)**

\[
|\langle Kg, g \rangle| \leq \eta |g|^{2}_{L^{2}_{\gamma+2s}(\mathbb{R}^{3})} + C_{\eta} |g|^{2}_{L^{2}(B_{C_{\eta}})}, \quad \forall \eta > 0, \quad \exists C_{\eta} > 0.
\]

**Theorem (Non-linear estimate)**

\[
|\langle \Gamma(g, h), f \rangle| \lesssim \left( |g|_{L^{2}} h_{N^{s, \gamma}} + |g|_{N^{s, \gamma}} |h|_{L^{2}} \right) |f|_{N^{s, \gamma}}.
\]
Advantage of our concrete definition $N^{s, \gamma}$: we can develop a corresponding Littlewood-Paley theory to prove the upper bounds:

$$P_j f \left( v, \frac{|v|^2}{2} \right) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dv' 2^{3j} \varphi \left( 2^j \left( v - v', \frac{|v|^2}{2} - \frac{|v'|^2}{2} \right) \right) \langle v' \rangle f(v')$$

$$Q_j f \overset{\text{def}}{=} P_j f - P_{j-1} f, \quad j \geq 1$$

This is a renormalized four-dimensional Euclidean Littlewood-Paley decomposition of the measure on the paraboloid in $\mathbb{R}^4$.

- We may phrase our results in terms of $P_j g$, $Q_j g$, and Euclidean derivatives of these functions in $\mathbb{R}^4$ instead of $\mathbb{R}^3$.
- We obtain the sharp characterization of the space $N^{s, \gamma}$:

$$\sum_{j=0}^{\infty} 2^{2(s-|\alpha|)j} \int_{\mathbb{R}^3} dv \ |\tilde{\nabla}^\alpha Q_j f(v)|^2 \langle v \rangle^{\gamma+2s} \lesssim |f|^2_{N^{s, \gamma}}$$

For any multi-index $\alpha$ of Euclidean derivatives $\tilde{\nabla}$ on $\mathbb{R}^4$. 

One idea for the estimates: Littlewood-Paley-Stein theory
Soft potential decay rates in the whole space: $\mathbb{R}^n_x$

**Theorem (Ukai-Asano 1982, with angular cut-off, $-1 < \gamma \leq 0$)**

\[
\| \langle v \rangle^b f(t, \cdot, \cdot) \|_{L^\infty(H^m_x)} \lesssim (1 + t)^{-a}.
\]

Here \(a \overset{\text{def}}{=} \min \left\{ \frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right) , 1 \right\} \) for \(p \in [1, 2)\). With

\[
\| \langle v \rangle^{b+a\gamma} f_0(\cdot, \cdot) \|_{L^\infty(H^m_x)} + \| f_0 \|_{L^2(L^p_x)} \text{ sufficiently small.}
\]

Above \(a = \frac{n}{4}\) when \(p = 1\) is optimal in dimensions \(n = 2, 3, 4\) but the rate of \(a = 1\) is not optimal for \(n \geq 5\).

**Theorem (AMUXY, 2011, non cut-off)**

*Optimal $L^2_x$ decay rates for the hard potentials $\gamma + 2s > 0$. For the soft potentials $\max\{-2s - 3/2, -3\} < \gamma < -2s$ they have*

\[
\text{ess sup}_{x \in \mathbb{R}^3_x} |f(t, x)| \|_{H^{K-3}(\mathbb{R}^3_x)} \lesssim (1 + t)^{-1}, \quad (K \geq 6).
\]

This $L^\infty_x$ rate is non-optimal.
We use the homogeneous mixed Besov space $\dot{B}_2^{-\varrho,\infty} L^2_v$ with norm

$$\|g\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} \overset{\text{def}}{=} \sup_{j \in \mathbb{Z}} \left(2^{-\varrho j} \|\Delta_j g\|_{L^2(\mathbb{R}^n_x \times \mathbb{R}^n_v)}\right), \quad \varrho > 0.$$ 

Here $\Delta_j$ is the standard LP projection onto frequencies $|\xi| \approx 2^j$.


*Fix $\varrho \in (0, n/2]$ and consider any $k \in \{0, 1, \ldots, K - 1\}$. Suppose additionally that $\|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$ (not necessarily small). Then*

$$\sum_{k \leq |\alpha| \leq K} \|\partial^\alpha f(t)\|_{L^2_x L^2_v}^2 \lesssim (1 + t)^{-(k+\varrho)}.$$ 

*This holds uniformly for $t \geq 0$.*

These decay rates are said to be “optimal” by comparison to the analogous results for the linear decay. Lower bounds unknown.
Theorem (Sohinger-S, same paper, faster decay)

Furthermore for \( \varrho \in (n/2, (n + 2)/2] \), if

\[
\| \mathcal{P} f_0 \\|_{\dot{B}^{n-\varrho, \infty}_2 L^2_v} + \| \{ I - \mathcal{P} \} f_0 \\|_{\dot{B}^{n+1-\varrho, \infty}_2 L^2_v} < \infty,
\]

then, in the hard potentials case, the solution uniformly satisfies

\[
\sum_{|\alpha| + |\beta| \leq K} \| \omega^{\ell - |\beta|} \partial_\beta^\alpha f(t) \|^2_{L^2_x L^2_v} \lesssim (1 + t)^{-\varrho},
\]

Recall the macroscopic projection is

\[
\mathcal{P} f = a^f(t, x)\sqrt{\mu} + \sum_{i=1}^n b_i^f(t, x)v_i\sqrt{\mu} + c^f(t, x)(|v|^2 - n)\sqrt{\mu}.
\]
Most previous results for decay rates in the whole space for these types of equations require that the initial data is small in $L^1_x$ (say).

- For the heat equation, if $g_0 \in \mathcal{S}'$ and $\|g_0\|_{\dot{B}_2^{-\varrho,\infty}(\mathbb{R}^n)} < \infty$ then

$$\|g_0\|_{\dot{B}_2^{-\varrho,\infty}(\mathbb{R}^n)} \approx \left\| t^{\varrho/2} \right\| e^{t\Delta} g_0 \right\|_{L^2(\mathbb{R}^n)} \left\| L^\infty_t((0,\infty)) \right\|, \text{ for any } \varrho > 0.$$

- For the incompressible Navier-Stokes equations, if $u_0 \in L_x^2 \cap \dot{B}_2^{-\varrho,\infty}$ generates a Leray-Hopf weak solution then

$$\sup_{t > 0} \left( t^{\varrho/2} \| u(t) \|_{L^2(\mathbb{R}^n)} \right) < \infty, \quad \varrho \in \left( 0, \frac{n + 2}{2} \right], \quad n \in \{2, 3\}.$$

Needed to establish vector valued interpolation.

To prove energy inequalities with high spatial derivatives, we need to do interpolation.

Our solution is to let $\mathcal{H}_v$ be an arbitrary separable Hilbert space, and re-prove vector valued interpolation inequalities.

\[
\|f\|_{\dot{B}^{-\varrho,\infty}_q \mathcal{H}_v} \lesssim \|f\|_{L^p_x \mathcal{H}_v}, \quad \frac{1}{p} - \frac{1}{q} = \frac{\varrho}{n}, \quad 1 \leq p \leq 2, \quad 0 < \varrho \leq \frac{n}{2}.
\]

\[
\|f\|_{\dot{B}^k,1_{\mathcal{H}_v}} \lesssim \|f\|_{\dot{B}^{1-\theta}_r \mathcal{H}_v} \|f\|_{\dot{B}^{\varrho,\infty}_r \mathcal{H}_v}, \quad 0 < \theta < 1, \quad 1 \leq r \leq p \leq \infty,
\]

Also $m \neq \varrho$ and $k + \frac{n}{r} - \frac{n}{p} = m(1 - \theta) + \varrho \theta$. These are in the “hard order” when the $\mathcal{H}_v$ norm is on the inside. And others...
Coercive Entropy production estimates

Boltzmann’s celebrated H-theorem:

\[-\frac{d}{dt} \int_{T^3} dx \int_{\mathbb{R}^3} dv \ F \log F = \int_{T^3} dx \ D(F) \geq 0.\]

- non-negative Entropy production functional:

\[D(F) = -\int_{\mathbb{R}^3} dv \ Q(F, F) \log F\]

\[= \frac{1}{4} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma \ B(\nu - \nu_*, \sigma) (F'F'_* - FF_*) \log \frac{F'F'_*}{FF_*}.\]

- Alexandre-Desvillettes-Villani-Wennberg (ARMA, 2000)

\[D(F) \gtrsim C_R \|\sqrt{F}\|_{H^s(|\nu|<R)}^2 - \text{l.o.t.}, \quad R > 0.\]

Widely used. General Conditions: \( F \in L \log L(\mathbb{R}^3) \cap L_1^1(\mathbb{R}^3). \)

\[ D(F) \gtrsim \int_{\mathbb{R}^3} dv \int_{d(v,v') \leq 1} dv' \langle v \rangle^{\gamma+2s+1} \left( \sqrt{F'} - \sqrt{F} \right)^2 \frac{d}{d(v,v')^{3+2s}} - \text{l.o.t.} \]

- This is the same semi-norm as in the linearized context.
- Stronger global anisotropic semi-norm (in terms of the weight power coupled to the fractional differentiation) than the ADVW (ARMA, 2000) smoothing estimate.
- Derivative estimate holds under general (local) conditions: \( F \in L \log L \cap L^1(B_R) \). (Plus other integrability conditions for the error terms which are the same as ADVW).
- Also work by Chen-He (2011), AMUXY, etc, in terms of sharp isotropic lower bound for this Sobolev space.
Congressman Jerry McNerney giving a “one-minute speech” on the floor of the US House of Representatives about our NSF-funded research on the Boltzmann equation on June 12, 2013.