$A_\infty$ structures as a language for open Gromov-Witten theory

Short talks by postdoctoral members, IAS, fall 2017
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Gromov-Witten theory \((g = 0)\)

**Setting:** \((X, \omega, J)\) symplectic manifold with almost complex structure

\[ X = X^{2n}, \ \omega \text{ 2-form such that } \omega^n \text{ is a volume form} \]
\[ J \in \text{End}(TX), J^2 = -\text{Id}, \text{ “}\omega\text{-tame”} \]

**Example:** \((\mathbb{C}P^n, \omega_{FS}, J_0)\)

**Problem:** Count \(J\)-holomorphic maps from the sphere

\[ u: S^2 \to X \]

that satisfy various constraints
The moduli space of sphere maps

\[ \overline{\mathcal{M}}_l(\beta) = \left\{ (u: S^2 \xrightarrow{\text{J-hol.}} X, w_1, \ldots, w_l) . \ [u] = \beta \in H_2(X; \mathbb{Z}) \mid w_j \in S^2, w_i \neq w_j \right\} / \sim \]

Compactification:
Rephrasing the problem

Count elements of $\overline{\mathcal{M}}_i(\beta)$ such that the marked points are mapped to given constraints.

Can be expressed as an integral:

$$GW_\beta(\gamma_1, \ldots, \gamma_l) = \int_{\overline{\mathcal{M}}_i(\beta)} ev_1^* \gamma_1 \wedge \cdots \wedge ev_l^* \gamma_l.$$
Some facts

• $GW$ invariants are defined by the above integral if the space $\overline{M}_i(\beta)$ is “nice”

• $GW$ are generally hard to compute

• In some cases, can compute $GW$ invariants by the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation
### Kontsevich (1994)

<table>
<thead>
<tr>
<th>degree = d</th>
<th>No. of degree-d curves in $\mathbb{C}P^2$ through 3d-1 points</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>26,312,976</td>
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<td>7</td>
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Open Gromov-Witten theory \((g = 0)\)

**Setting:** \((X, \omega, J)\) symplectic manifold with almost complex structure

\[ L \subset X \text{ a Lagrangian submanifold } \quad (\dim L = \frac{1}{2} \dim X, \omega|_L = 0) \]

**Example:** \((X, L, \omega, J) = (\mathbb{C}P^n, \mathbb{R}P^n, \omega_{FS}, J_0)\)

**Problem:** Count \(J\)-holomorphic maps from the disk

\[ u: (D, \partial D) \to (X, L) \]

that satisfy various constraints.
The moduli space of disk maps

$$\overline{\mathcal{M}}_{k,l}(\beta) = \left\{ \left( u: (D, \partial D) \xrightarrow{\text{j-hol.}} (X, L), z_1, \ldots, z_k, w_1, \ldots, w_l \right) : [u] = \beta \in H_2(X, L; \mathbb{Z}) \right\} / \sim$$

Compactification:
Rephrasing the problem

Count elements of $\mathcal{M}_{k,l}(\beta)$ such that the marked points are mapped to given constraints

Can be expressed as an integral:

$$OGW_\beta(\alpha_1, \ldots, \alpha_k; \gamma_1, \ldots \gamma_l) =$$

$$= \int_{\mathcal{M}_{k,l}(\beta)} evb_1^* \alpha_1 \wedge \cdots \wedge evb_k^* \alpha_k \wedge evi_1^* \gamma_1 \wedge \cdots \wedge evi_l^* \gamma_l.$$

**Issue:** $\partial \mathcal{M}_{k,l}(\beta) \neq \emptyset$. 
Some previous results

**OGW are defined when**

• $S^1$ acts on $(X, L)$ \((Liu, 2004)\)
• $(X, L, \omega, J)$ is a real symplectic manifold with $\dim_{\mathbb{C}} X = 2,3$, real interior constraints, point boundary constraints \((Cho, Solomon, 2006)\)
• $(X, L, \omega, J)$ is a real symplectic manifold with $\dim_{\mathbb{C}} X$ odd, no boundary constraints \((Georgieva, 2013)\)

**OGW are computable via a WDVV-like equation when**

• $(X, L, \omega, J)$ is a real symplectic manifold with $\dim_{\mathbb{C}} X = 2$, real interior constraints, point boundary constraints \((Horev-Solomon, 2012)\)
• $(X, L, \omega, J)$ is a real symplectic manifold with $\dim_{\mathbb{C}} X$ odd, no boundary constraints \((Georgieva-Zinger, 2013)\)
\( A_\infty \) structure

= Algebraic language to describe boundary behavior

- \( A_\infty \) operators describe disks with prescribed boundary constraints
- \( A_\infty \) relations describe disk bubbling

\[ \partial \left( \{ \ldots \} \right) = \{ \ldots \} + \{ \ldots \} \]

- Special kind of boundary constraint: “bounding chain”
More results (joint with Jake Solomon)

• \( OGW \) can be defined using bounding chains when \( \dim_\mathbb{C} X \) is odd, under cohomological conditions. E.g., \( H^* (L; \mathbb{R}) = H^* (S^n; \mathbb{R}) \).

• The boundary constraints can be interpreted as points.

• Whenever defined, \( OGW \) satisfy open WDVV equations.

• For \( (\mathbb{C}P^n, \mathbb{R}P^n) \), all invariants are determined by the open WDVV.
\((X, L) = (\mathbb{CP}^n, \mathbb{RP}^n)\)

Initial condition: \(OGW_{1,2}^n = 2\)

<table>
<thead>
<tr>
<th>dim = n</th>
<th>degree = d</th>
<th>No. of boundary points = k</th>
<th>Resulting invariant (OGW_{d,k}^n)</th>
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</table>
More questions

• Reduce cohomological assumptions
• Find structure suitable for $g > 0$

• Explore relative quantum cohomology

open WDVV $\leftrightarrow$ associativity of relative quantum product
Thank you