Floer theory and metrics in symplectic and contact topology

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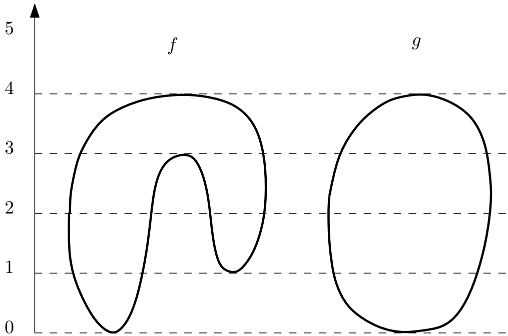
September 27, 2016

Changing geometry/topology requires energy



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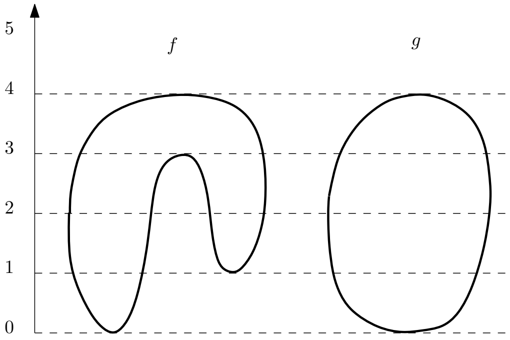

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Otherwise $|f - g \circ \psi|_{C^0} = 1 - \epsilon$,

$$\{f < 1 + \epsilon\} \subset \{g \circ \psi < 2\} \subset \{f < 3 - \epsilon\}$$

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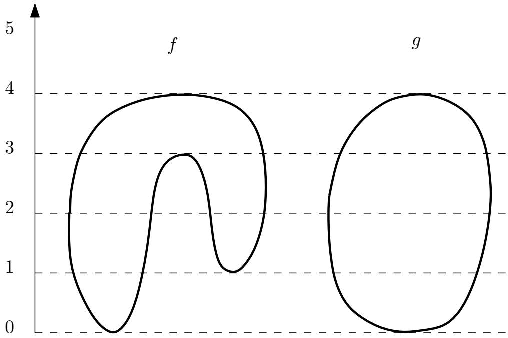

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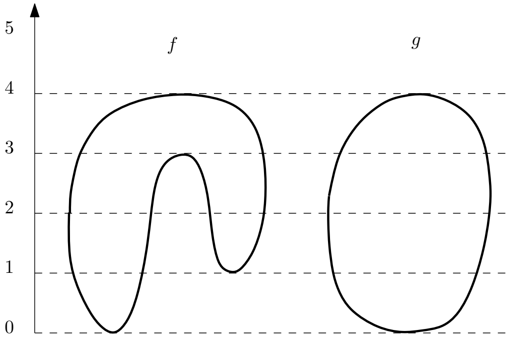

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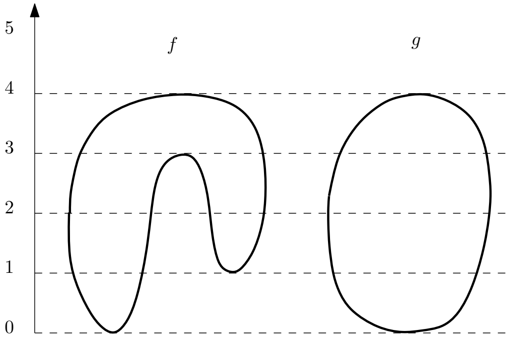

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composition iso:

$$2 = \dim V^{1+\epsilon}(f) \le \dim V^2(g) = 1.$$

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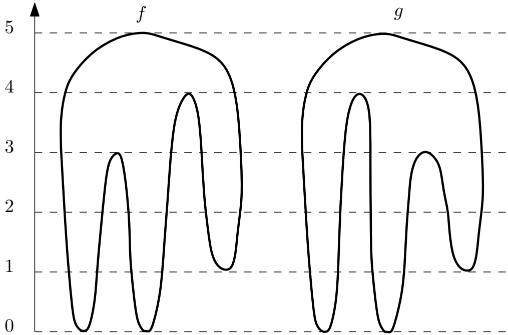

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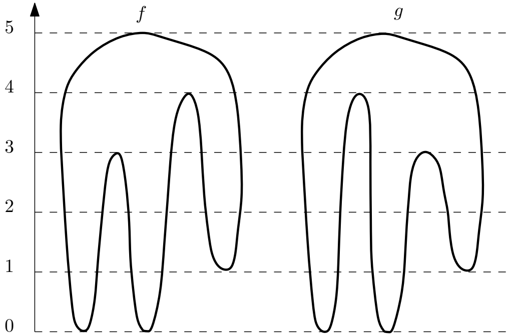

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Not so easy. Dimensions are the same!

Hamiltonian diffeomorphisms

- (M, ω) - closed symplectic manifold.
- $Ham := Ham(M, \omega)$ - the group of Hamiltonian diffeomorphisms:

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  $\iota_{X_H^t}\omega = -d(H(t, -)) \quad H \in C^\infty([0,1] \times M, \mathbb{R})$
- universal cover $\widetilde{Ham} = \{\{\phi_H^t\}_{t=0}^1 | \, H \, ...\} / \sim$,
  $\sim$ is homotopy with fixed endpoints.

Metrics on groups

G - group.

A metric d on G - *right-invariant*: $d(ag, bg) = d(a, b)$ for all $a, b, g \in G$.

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Hofer norm

Definition

$$d_{\mathrm{Hofer}}(f, g) = \inf_{H:\phi_H^1 = gf^{-1}} \int_0^1 \left(\max_M H(t, -) - \min_M H(t, -)\right) dt.$$

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Remark

- With normalization $\int_M H(t,-)\,\omega^n = 0$, can take $|H(t,-)|_{L^\infty(M)} = \max_M |H(t,-)| \rightsquigarrow$ equivalent metric (cf. Bukhovsky-Ostrover).
- In contrast: false for $|H(t,-)|_{L^p}$ (Eliashberg-Polt.), no fine conj. invt. norms on $Diff_0$, $Cont_0$ (Burago-Ivanov-Polt.,Fraser-Polt.-Rosen).

Filtered Floer homology

"Morse theory for action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$, *for* H *- Hamiltonian"*

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$\rightsquigarrow$ *(pointwise fin. dim. constructible) persistence module*

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Can show: dep. only on ϕ_H^1 (more generally on $[\{\phi_H^t\}]$)

$\rightsquigarrow V^a(\phi)_*$ persistence module.

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The multiset of intervals is canonical $\rightsquigarrow$ "barcode".

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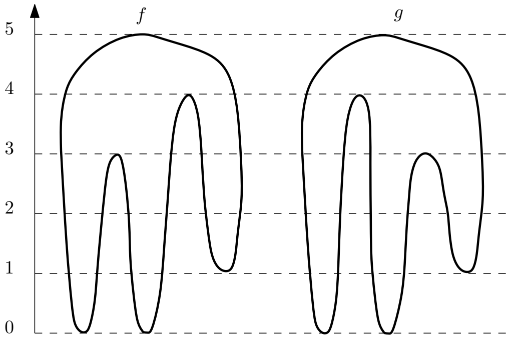
f and g: one answer



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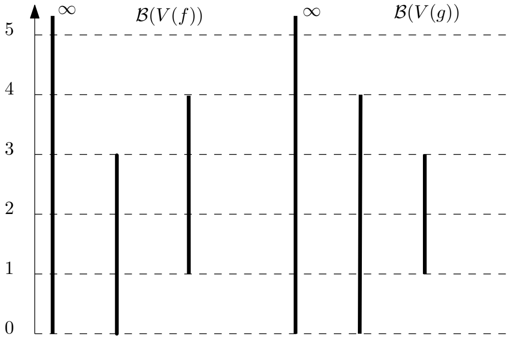

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Theorem

(Polterovich-S., 2015) In $G = Ham(\Sigma_4 \times N)$ *exist* ϕ_j *s.t.*

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(Polterovich-S.-Stojisavljevic, in progress) Same for $G = Ham(\Sigma_4 \times \mathbb{C}P^n)$.

Uses action of quantum homology on Floer persistence.
(Zhang - same for any M, where power p-large)

Lagrangian submanifolds

$L \subset M$ closed Lagrangian submanifold,
$\dim L = \frac{1}{2} \dim M, \omega|_L = 0$. Assume $\pi_2(M, L) = 0$. Then
$HF(L, L) \cong H(L)$.

If $\phi \in Ham$, $\phi L \neq L$, then $d_{\mathrm{Hofer}}(\phi, 1) > 0$. (Chekanov, Barraud-Cornea, Charette,...)

Proof:

- barcodes of $HF(L, \phi L)^t$, $HF(L, L)^t$ are at distance at most $d_{\mathrm{Hofer}}(\phi, 1)$. Hence $\dim H(L)$ inifinite bars which start below $d_{\mathrm{Hofer}}(\phi, 1)$
- so are $pt * HF(L, \phi L)^t, pt * HF(L, L)^t$, hence one infinite bar that starts above $-d_{\mathrm{Hofer}}(\phi, 1)$.
- hence exist $x, y \in CF(L, \phi L)$ with $pt * x = y$ and $\mathcal{A}(x) - \mathcal{A}(y) \leq 2d_{\mathrm{Hofer}}(\phi, 1)$.

- A bit of Gromov compactness shows that via any point $p \in L \setminus \phi L$ and any J, there is a J-holomorphic strip with $Area \leq 2d_{\mathrm{Hofer}}(\phi, 1)$
- choosing good J, and standard monotonicity argument: $Area \geq \pi r^2/2$, with $B(r)$ standard symplectic ball of radius r, embedded (only) with real part on L, and disjoint from L'.
- $d_{\mathrm{Hofer}}(\phi, 1) \geq \pi r^2/4$

(Cornea-S., 2015) Generalize to certain Lagragian cobordisms
(Biran-Cornea-S., in progress) Generalize to multi-ended cobordisms, and isomorphisms in the Fukaya category.

Contactomorphisms

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pf of some rigidity statement $\rightsquigarrow$ pf with hol. curves $\rightsquigarrow$ new methods, new results.

Contactomorphisms

Theorem

(Givental, 1990, using fin.-dim. methods):
$\exists \nu_0 : \widetilde{\mathrm{Cont}}_0(\mathbb{R}P^{2n+1}, \xi_{st}) \to \mathbb{R}$ *unbounded quasi-morphism*
$\sup_{x,y} |\nu_0(xy) - \nu_0(x) - \nu_0(y)| < \infty$.

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Question

Are these two related?

Answer: no idea, but

Theorem

(Ben-Simon, 2006) $i^*\nu_0$ *has the Calabi property, like* μ, *where* $i : \widetilde{Ham}(\mathbb{C}P^n) \to \widetilde{\mathrm{Cont}}_0(\mathbb{R}P^{2n+1})$ *natural inclusion.*

Theorem

(Albers-S.-Zapolsky, in progress, using Floer theory): $\exists \nu : \widetilde{\mathrm{Cont}}_0(\mathbb{R}P^{2n+1}, \xi_{st}) \to \mathbb{R}$ *unbounded quasi-morphism, for which*

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Idea: use package of filtered Lagragian Floer homology for $\mathbb{R}P^{2n+1} \hookrightarrow S(\mathbb{R}P^{2n+1}) \times (\mathbb{C}P^n)^-$, Lagrangian correspondence. Obstacle: concave end. Upshot: new results e.g. on topology of Cont.

Thank you!