Floer theory and metrics in symplectic and contact topology

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Changing geometry/topology requires energy

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Otherwise \(|f - g \circ \psi|_{C^0} = 1 - \epsilon,\)

\(\{f < 1 + \epsilon\} \subset \{g \circ \psi < 2\} \subset \{f < 3 - \epsilon\}\)
Changing geometry/topology requires energy

\[ V^t(f) = H_0(\{ f < t \}, \mathbb{K}), \quad V^t(g) = V^t(g \circ \psi) \]
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composition iso:

\[ 2 = \dim V^{1+\epsilon}(f) \leq \dim V^2(g) = 1. \]
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Not so easy. Dimensions are the same!
Hamiltonian diffeomorphisms

- $(M, \omega)$ - closed symplectic manifold.

- $\text{Ham} : = \text{Ham}(M, \omega)$ - the group of Hamiltonian diffeomorphisms:

  endpoints of paths $\{\phi^t_H\}_{t=0}^1 \quad \phi^0_H = id$
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  generated by vector field \(X^t_H\)

  \(\iota_{X^t_H} \omega = -d(H(t, -))\) \(H \in C^\infty([0, 1] \times M, \mathbb{R})\)
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  $\iota_{X^t_H}\omega = -d(H(t, -)) \quad H \in C^\infty([0, 1] \times M, \mathbb{R})$

- universal cover $\widehat{\text{Ham}} = \{(\{\phi^t_H\}_{t=0}^1 | H \ldots)/ \sim\}, \sim$ is homotopy with fixed endpoints.
Metrics on groups

$G$ - group.

A metric $d$ on $G$ - right-invariant: $d(ag, bg) = d(a, b)$ for all $a, b, g \in G$.

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Hofer norm

**Definition**

\[ d_{\text{Hofer}}(f, g) = \inf_{H: \phi_H^1 = gf^{-1}} \int_0^1 (\max_M H(t, -) - \min_M H(t, -)) \, dt. \]
Hofer norm

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\[ d_{\text{Hofer}}(f, g) = \inf_{H: \phi_H^1 = gf^{-1}} \int_0^1 \left( \max_M H(t, -) - \min_M H(t, -) \right) dt. \]

Theorem

(Hofer, Viterbo, Polterovich, Lalonde-McDuff)

\( d_{\text{Hofer}} \) is a bi-invariant metric on \( \text{Ham} \).
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Theorem

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\( d_{\text{Hofer}} \) is a bi-invariant metric on Ham.

Remark

▶ With normalization \( \int_M H(t, -) \omega^n = 0 \), can take
\[ |H(t, -)|_{L^\infty(M)} = \max_M |H(t, -)| \rightsquigarrow \) equivalent metric
(cf. Bukhovsky-Ostrover).

▶ In contrast: false for \( |H(t, -)|_{L^p} \) (Eliashberg-Polt.), no
fine conj. invt. norms on \( \text{Diff}_0, \text{Cont}_0 \)
(Burago-Ivanov-Polt., Fraser-Polt.-Rosen).
"Morse theory for action functional $A_H : \mathcal{LM} \to \mathbb{R}$, for $H$ - Hamiltonian”

$A_H(z) = \int_0^1 H(t, z(t)) \, dt - \int_z \omega$
Filtered Floer homology

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$\text{Crit}(A_H) = 1$-periodic orbits of $\{\phi^t_H\}$. 
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For $f$ - Morse on closed mfld,

$V^a(f)_* = H_*(\{f < a\})$. 
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Triangles for $a \leq b \leq c$ commute
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"Morse theory for action functional \( A_H : \mathcal{L}M \to \mathbb{R}, \) for \( H \) - Hamiltonian"

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\( \rightsquigarrow \text{(pointwise fin. dim. constructible) persistence module} \)
Similarly, assuming $[\omega]|_{tori} = 0$, $[c_1]|_{tori} = 0$ (otherwise need to work with coeff. in ”Novikov ring”),

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\[ H \text{ with } \text{graph}(\phi^1_H) \cap \Delta, \rightsquigarrow V^a(H)_\ast = "HF(-\infty,a)(H)_\ast" \]

Can show: dep. only on $\phi^1_H$ (more generally on $[\{\phi^t_H\}]$)

$\rightsquigarrow V^a(\phi)_\ast$ persistence module.
Persistence

\[ I = (a, b] \text{ or } (a, \infty) - \text{ interval}; \]
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Interval p-mod: \( Q(I) \) with \( Q(I)^a = \mathbb{K} \) iff \( a \in I \), otherwise 0. \( (\pi^{a,b} \text{ iso whenever can}) \)
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Theorem
(Carlsson-Zomorodian, Crawley-Boevey) Every p.-mod. as above is isomorphic to a finite direct sum of interval p-modules.
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The multiset of intervals is canonical \( \rightsquigarrow \) ”barcode”.
Isometry theorem

(Edelsbrunner - Harer - Cohen-Steiner,...,Bauer-Lesnick) ⇒

If $|f - g|_{C^0} \leq C$ then barcodes of $V(f), V(g)$ are related by "moving endpts of bars $\leq C$".

Similar conclusion of $V(\varphi), V(\psi)$ for $d_{Hofer}(\varphi, \psi) \leq C$.

⇒ "length of maximal bar" (= boundary depth - Usher), "max starting pt inf. bar" (= fund. class spectral invt - Viterbo, Oh, Schwarz,...) etc. are Lipschitz in Hofer's metric.
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If \|f - g\|_{C^0} \leq C

⇒ 

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\(\Rightarrow\) "length of maximal bar" (\(=\) boundary depth - Usher),
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\[ \inf_{\psi \in \text{Diff}_0(S^1)} |f - g \circ \psi| = 1 \]

since

\[ d(\mathcal{B}(V(f)), \mathcal{B}(V(g))) \geq 1 \]
f and g: one answer

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Far from a power

Theorem
(Polterovich-S., 2015) In $G = \text{Ham}(\Sigma_4 \times N)$ exist $\phi_j$ s.t.

$$d_{\text{Hofer}}(\phi_j, \{\theta^2 | \theta \in G\}) \xrightarrow{j \to \infty} \infty,$$

$N$ symp. aspherical or a point.
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**Question**

Same for $\text{Ham}(S^2)$? Even for $\text{im}(\exp)$?
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Theorem
(Polterovich-S.-Stojisavljevic, in progress) Same for $G = \text{Ham}(\Sigma_4 \times \mathbb{C}P^n)$.

Uses action of quantum homology on Floer persistence.
(Zhang - same for any $M$, where power $p$-large)
Lagrangian submanifolds

\( L \subset M \) closed Lagrangian submanifold, 
\( \dim L = \frac{1}{2} \dim M, \omega|_L = 0. \) Assume \( \pi_2(M, L) = 0. \) Then 
\( HF(L, L) \cong H(L). \)

If \( \phi \in Ham, \phi L \neq L, \) then \( d_{\text{Hofer}}(\phi, 1) > 0. \) (Chekanov, Barraud-Cornea, Charette,...)

Proof:

- barcodes of \( HF(L, \phi L)^t, HF(L, L)^t \) are at distance at most \( d_{\text{Hofer}}(\phi, 1) \). Hence \( \dim H(L) \) infinite bars which start below \( d_{\text{Hofer}}(\phi, 1) \)
- so are \( pt \ast HF(L, \phi L)^t, pt \ast HF(L, L)^t \), hence one infinite bar that starts above \( -d_{\text{Hofer}}(\phi, 1) \).
- hence exist \( x, y \in CF(L, \phi L) \) with \( pt \ast x = y \) and 
\( A(x) - A(y) \leq 2d_{\text{Hofer}}(\phi, 1). \)
A bit of Gromov compactness shows that via any point $p \in L \setminus \phi L$ and any $J$, there is a $J$-holomorphic strip with $\text{Area} \leq 2d_{\text{Hofer}}(\phi, 1)$.

Choosing good $J$, and standard monotonicity argument: $\text{Area} \geq \pi r^2/2$, with $B(r)$ standard symplectic ball of radius $r$, embedded (only) with real part on $L$, and disjoint from $L'$.

$d_{\text{Hofer}}(\phi, 1) \geq \pi r^2/4$

(Cornea-S., 2015) Generalize to certain Lagragian cobordisms
(Biran-Cornea-S., in progress) Generalize to multi-ended cobordisms, and isomorphisms in the Fukaya category.
Contactomorphisms

Eliashberg’s dichotomy, 2014:

"Holomorphic curves or nothing" or: something ⇒ holomorphic curves! 

pf of some rigidity statement ⇝ pf with hol. curves ⇝ new methods, new results.
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something $\Rightarrow$ holomorphic curves!

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Theorem

(Givental, 1990, using fin.-dim. methods):
\[ \exists \nu_0 : \widehat{\text{Cont}}_0(\mathbb{R}P^{2n+1}, \xi_{st}) \to \mathbb{R} \text{ unbounded quasi-morphism} \]
\[ \sup_{x,y} |\nu_0(xy) - \nu_0(x) - \nu_0(y)| < \infty. \]
Contactomorphisms

Theorem
(Givental, 1990, using fin.-dim. methods):
\[ \exists \nu_0 : \widetilde{\text{Cont}}_0(\mathbb{RP}^{2n+1}, \xi_{st}) \to \mathbb{R} \text{ unbounded quasi-morphism} \]
\[ \sup_{x,y} |\nu_0(xy) - \nu_0(x) - \nu_0(y)| < \infty. \]

Theorem
(Entov-Polterovich, 2003, using Floer theory): same for \[ \mu : \widetilde{\text{Ham}}(\mathbb{CP}^n, \omega_{st}) \to \mathbb{R}. \]
Contactomorphisms

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**Theorem**

*(Entov-Polterovich, 2003, using Floer theory):* same for

\[ \mu : \widetilde{\text{Ham}}(\mathbb{C}P^n, \omega_{st}) \to \mathbb{R}. \]

**Question**

Are these two related?
Answer: no idea, but

**Theorem**

(Ben-Simon, 2006) $i^* \nu_0$ has the Calabi property, like $\mu$, where $i : \widehat{\text{Ham}}(\mathbb{C}P^n) \to \widehat{\text{Cont}}_0(\mathbb{R}P^{2n+1})$ natural inclusion.

**Theorem**

(Albers-S.-Zapolsky, in progress, using Floer theory):

$\exists \nu : \widehat{\text{Cont}}_0(\mathbb{R}P^{2n+1}, \xi_{st}) \to \mathbb{R}$ unbounded quasi-morphism, for which

$$i^* \nu = \mu.$$
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\[ i^* \nu = \mu. \]

Idea: use package of filtered Lagragian Floer homology for $\mathbb{R}P^{2n+1} \hookrightarrow S(\mathbb{R}P^{2n+1}) \times (\mathbb{C}P^n)^-$, Lagrangian correspondence.
Obstacle: concave end. Upshot: new results e.g. on topology of Cont.
Thank you!