Universal approach to $\beta$-matrix models

M. Shcherbina

Institute for Low Temperature Physics, Ukr. Ac. Sci.
Kharkov, Ukraine

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Model definition

Distributions in $\mathbb{R}^n$, depending on the function $V$ and $\beta > 0$

$$p_{n, \beta}(\lambda_1, \ldots, \lambda_n) = Z_n^{-1}[\beta, V]e^{\beta H(\lambda_1, \ldots, \lambda_n)/2},$$

where $H$ (Hamiltonian) and $Z_n[\beta, V]$ (partition function) are

$$H(\lambda_1, \ldots, \lambda_n) = -n \sum_{i=1}^{n} V(\lambda_i) + \sum_{i \neq j} \log |\lambda_i - \lambda_j|,$$

$$Z_n[\beta, V] = \int e^{\beta H(\lambda_1, \ldots, \lambda_n)/2}d\lambda_1 \ldots d\lambda_n,$$

$$V(\lambda) > (1 + \varepsilon) \log(1 + \lambda^2).$$

For $\beta = 1, 2, 4$ it is a joint eigenvalues distribution of real symmetric, hermitian and symplectic matrix models respectively.
Expectation and correlation functions

For given $h : \mathbb{R}^n \to \mathbb{C}$, \[ \langle h \rangle_{V,n} = \int h(\lambda_1, \ldots, \lambda_n) p_{n,\beta}(\lambda_1, \ldots, \lambda_n) d\bar{\lambda} \]

Correlation functions (marginal densities):

\[ p_{n,\beta}^{(m)}(\lambda_1, \ldots, \lambda_m) = \int_{\mathbb{R}^{n-1}} p_{n,\beta}(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_n) d\lambda_{m+1} \ldots d\lambda_n \]

The linear eigenvalue statistics (LES) and the counting measure of eigenvalues

\[ N_n[h] = \sum_{j=1}^{n} h(\lambda_j), \quad N_n[\Delta] = \sum_{j=1}^{n} 1_\Delta(\lambda_j). \]
Main problems of the global regime

1. weak limit of the first correlation function $w - \lim_{n \to \infty} p_{n, \beta}^{(1)}(\lambda) = \rho(\lambda)$, support $\sigma$ of $\rho(\lambda)$;

2. weak limits of the other correlation functions $p_{n, \beta}^{(m)}(\lambda_1, \ldots, \lambda_m)$ and their factorization property

$$p_{n, \beta}^{(m)}(\lambda_1, \ldots, \lambda_m) - p_{n, \beta}^{(1)}(\lambda_1) \ldots p_{n, \beta}^{(1)}(\lambda_m) \to 0, \quad \text{as } n \to \infty;$$

3. large deviation type bounds for the correlation functions;

4. generating functional of LES

$$\Phi[t, h] = \langle e^{\beta t N_n[h]/2} \rangle_{V, n} = \frac{Z_n[\beta, V - \frac{1}{n} h]}{Z_n[\beta, V]}$$

and CLT for LES.

5. expansion in $n^{-1}$ for $\log Z_n[\beta, V]$ and correlation functions;
Main problems of the local regime

1. Universality of local eigenvalue statistics. In the bulk case it means that for any \( \lambda_0 \in \sigma (\rho(\lambda_0) \neq 0) \) all correlation functions after a proper scaling have limits which do not depend on \( V \), i.e. the limits

\[
\lim_{n \to \infty} (\rho(\lambda_0))^{-m} p_{n,\beta}^{(m)}(\lambda_0 + s_1/n\rho(\lambda_0), \ldots, \lambda_0 + s_m/n\rho(\lambda_0))
\]

coincide with that for the Gaussian case \( V^*(\lambda) = \frac{1}{2} \lambda^2 \).

2. Universality of gap probabilities. For a fixed system of nonintersecting intervals \( \bar{\Delta} = (\Delta_1, \ldots, \Delta_k) \) and \( \bar{m} = (m_1, \ldots, m_k) \) introduce the indicators functions

\[
\psi_{\bar{\Delta},\bar{m}}(\bar{\lambda}; \lambda_0) := \prod_{j=1}^{n} \left( 1 - \phi(n\rho(\lambda_0)(\lambda_j - \lambda_0)) \right), \quad 0 \leq \phi(x) \leq 1, \quad |\text{supp } \phi| < \infty.
\]

Universality means that

\[
\lim_{n \to \infty} \langle \psi_{\bar{\Delta},\bar{m}}(\bar{\lambda}; \lambda_0) \rangle_{V,n} = \lim_{n \to \infty} \langle \psi_{\bar{\Delta},\bar{m}}(\bar{\lambda}; 0) \rangle_{*,n}
\]

3. Universality of the generating functional, which has the form

\[
\psi_\phi(\bar{\lambda}; \lambda_0) := \prod_{j=1}^{n} \left( 1 - \phi(n\rho(\lambda_0)(\lambda_j - \lambda_0)) \right), \quad 0 \leq \phi(x) \leq 1, \quad |\text{supp } \phi| < \infty.
\]
The equilibrium problem

\[ \mathcal{E}[V] = - \min_{m \in \mathcal{M}_1} \left\{ -L[dm, dm] + \int V(\lambda)m(d\lambda) \right\} = \mathcal{E}_V(m^*), \]

where \( L[dm, dm'] = \int \log |\lambda - \mu|dm(\lambda)dm'(\mu), \)

For any continuous \( V \) the problem has a unique solution \( m^* \). If \( V' \) is a Hölder function then \( m^*(d\lambda) \) has the density \( m^*(d\lambda) = \rho(\lambda)d\lambda \) with a compact support \( \sigma := \text{supp } m^* \). The density \( \rho \) is an equilibrium density and it is uniquely defined by the condition

\[ v(\lambda) := 2 \int \log |\lambda - \mu|\rho(\mu)d\mu - V(\lambda) = v^* = \text{const}, \quad \lambda \in \sigma \]

\[ v(\lambda) \leq v^*, \quad \lambda \notin \sigma \]

Without loss of generality we can assume that \( v^* = 0 \).
The first step for the global regime

**Theorem [Boutet de Monvel, Pastur, S:95; Johansson:98]**

If $V$ is a Hölder function, then

$$\log Z_n[\beta, V] = \frac{n^2 \beta}{2} \mathcal{E}[V] + O(n \log n),$$

where $\mathcal{E}[V] = \mathcal{E}_V(m^*)$.

Moreover, if $h' \in L_2[\sigma_\varepsilon]$

$$|n^{-1}E\{\mathcal{N}_n[h]\} - (h, m^*)| \leq Cn^{-1/2} \log^{1/2} n ||h'||_{2}^{1/2} ||h||_{2}^{1/2}$$
Small perturbations for one cut potentials

**Theorem [Johansson:98]**

If $V$ is a polynomial, $\sigma = [-2,2]$, and $\rho$ is "generic", $h : \mathbb{R} \to \mathbb{R}$ with $||h^{(6)}||_\infty, ||h'||_\infty \leq \epsilon n^{1/3}$, $\dot{h} := h - (\rho, h)$

$$
\langle e^{\beta N_n[h]/2} \rangle_{V,n} = \exp \left\{ (1 - \frac{\beta}{2}) (h, \nu) + \frac{\beta}{8} (\overline{D}_\sigma h, h) \right\} \left( 1 + n^{-1} O (||h^{(4)}||_\infty^3) \right)
$$

where the "variance operator" $\overline{D}_\sigma$ depends only of $\sigma$, and the measure $\nu$ have the form

$$(h, \nu) := \frac{1}{4} (h(-2) + h(2)) - \frac{1}{2\pi} \int_\sigma \frac{h(\lambda)d\lambda}{\sqrt{4 - \lambda^2}} + \frac{1}{2} (D_\sigma \log P, h)
$$

$P$ is defined by the relation $\rho(\lambda) = (2\pi)^{-1} P(\lambda) \sqrt{4 - \lambda^2}$

**Remark**

$D_\sigma$ is a rank one perturbation of $-\mathcal{L}_\sigma^{-1}$, where $\mathcal{L}_\sigma$ is the integral operator defined by the kernel $\log |\lambda - \mu|$ for the interval $\sigma$.
Large deviation type bounds

Take any n-independent small $\varepsilon > 0$. It was proven in [Albeverio, Pastur, S:01] that if we replace in the definition of the partition function and of the correlation functions the integration over $\mathbb{R}$ by the integration $\sigma_\varepsilon$, then $p_{n,\beta}^{(m)}$ and the new marginal densities $p_{n,\beta}^{(m,\varepsilon)}$ for $m = 1, 2, \ldots$ satisfy the inequalities

$$
\sup_{\lambda_1, \ldots, \lambda_m \in \sigma_\varepsilon} |p_{n,\beta}^{(m)}(\lambda_1, \ldots, \lambda_m) - p_{k,\beta}^{(m,\varepsilon)}(\lambda_1, \ldots, \lambda_m)| \leq C_m e^{-n\beta d\varepsilon},
$$

$$Z_n[\beta, V] = Z_n^{(\varepsilon)}[\beta, V](1 + e^{-n\beta d\varepsilon}).$$

It is more convenient to consider the integration with respect to $\sigma_\varepsilon$, thus, starting from this moment it is assumed that this truncation is made, and below the integration without limits means the integration over $\sigma_\varepsilon$, but the superindex $\varepsilon$ will be omitted.
Change of variables in the one cut case

Let $V$ be some smooth enough potential with equilibrium density $\rho$ such that $\text{supp}\rho = [-2, 2]$, and $\zeta(\lambda): \sigma_\varepsilon = [-2 - \varepsilon, 2 + \varepsilon] \to \sigma_\varepsilon$ be some smooth function such that $\inf_{\sigma_\varepsilon} \zeta' > 0$.
Consider

$$H^{(\zeta)}(\lambda_1, \ldots, \lambda_n) = -n \sum V(\zeta(\lambda_j)) + \sum_{i \neq j} \log |\zeta(\lambda_i) - \zeta(\lambda_j)| + \frac{2}{\beta} \sum \log \zeta'(\lambda_j)$$

It is evident that the corresponding partition function and all the marginal densities satisfy the relations

$$Z^{(\zeta)}_{n, \beta} := \int e^{\beta H^{(\zeta)}/2} d\bar{\lambda} = Z_n[\beta, V]$$

$$p^{(m, \zeta)}_{n, \beta}(\lambda_1, \ldots, \lambda_m) := (Z^{(\zeta)}_{n, \beta})^{-1} \int e^{\beta H^{(\zeta)}/2} d\lambda_{m+1} \ldots d\lambda_n = p^{(m)}_{n, \beta}(\zeta(\lambda_1), \ldots, \zeta(\lambda_m))$$
On the other hand,

\[ H^{(\zeta)}(\lambda_1, \ldots, \lambda_n) = -n \sum V(\zeta(\lambda_j)) + \sum_{i \neq j} \log |\lambda_i - \lambda_j| \]

\[ + \sum_{i,j} \log \left| \frac{\zeta(\lambda_i) - \zeta(\lambda_j)}{\lambda_i - \lambda_j} \right| + \left( \frac{2}{\beta} - 1 \right) \sum \log \zeta'(\lambda_j) \]

Denote

\[ L^{(\zeta)}(\lambda, \mu) := \log \left| \frac{\zeta(\lambda) - \zeta(\mu)}{\lambda - \mu} \right| = L^{(\zeta)}_+(\lambda, \mu) - L^{(\zeta)}_-(\lambda, \mu) = \sum \eta_k \psi_k(\lambda) \psi_k(\mu), \]

where \( L^{(\zeta)}_+ \) and \( L^{(\zeta)}_- \) are positive compact operators in \( L_2[\mathbb{R}] \) having smooth kernels (there is some freedom here which we will be used below).

For sufficiently smooth \( \zeta(\lambda) \) these operators have smooth eigenfunctions \( \{\psi_{k\pm}(\lambda)\}_{k=1}^{\infty} \) and eigenvalues \( \{\eta_{k\pm}\}_{k=1}^{\infty} \) such that if we denote \( \psi_{2k-1}(\lambda) := \psi_{k+}(\lambda), \psi_{2k}(\lambda) := \psi_{k-}(\lambda) \) and \( \eta_{2k-1} = \eta_{k+}, \eta_{2k} = \eta_{k-} \) the convergence above is uniform in \( \sigma_\varepsilon \)
Choice of $\zeta(\lambda)$

Choose $\zeta(\lambda)$ from the equation

$$\zeta'(\lambda) = \frac{\rho_{sc}(\lambda)}{\rho(\zeta(\lambda))}, \quad \zeta(-2) = -2,$$

where

$$\rho_{sc}(\lambda) = (2\pi)^{-1} \sqrt{4 - \lambda^2},$$

and

$$\rho(\lambda) = (2\pi)^{-1} P(\lambda) \sqrt{4 - \lambda^2}$$

is the equilibrium density corresponding to $V$.

Then

$$\zeta(2) = 2 \quad \rho(\zeta(\lambda))\zeta'(\lambda) = \rho_{sc}(\lambda)$$

and $\zeta(\lambda)$ could be extended to $\sigma_{\varepsilon}$ with the same number of derivatives as $P$. 
For this choice of ζ write

\[
\sum_{i,j} L^{(ζ)}(λ_i, λ_j) = \sum_k \eta_k \left( \sum_j ψ_k(λ_i) \right)^2 = \sum_k \eta_k \left( \sum_j (ψ_k(λ_j) - (ψ_k, ρ_{sc})) \right)^2
\]

\[
+ 2n \sum_j \sum_k \eta_k ψ_k(λ_j)(ψ_k, ρ_{sc}) - n^2 \sum_k \eta_k (ψ_k, ρ_{sc})^2
\]

\[
= R(\bar{λ}) + 2n \sum_j \int L^{(ζ)}(λ_j, μ)ρ_{sc}(μ)dμ - n^2 \int L^{(ζ)}(λ, μ)ρ_{sc}(λ)ρ_{sc}(μ)dλdμ
\]

where \((f, g) := \int fgdλ\). It is easy to see that

\[
2 \int L^{(ζ)}(λ_j, μ)ρ_{sc}(μ)dμ = V(ζ(λ_j)) - \frac{λ_j^2}{2},
\]

\[
\int L^{(ζ)}(λ, μ)ρ_{sc}(λ)ρ_{sc}(μ)dλdμ = \mathcal{E}_{sc} - \mathcal{E}_V =: -Δ\mathcal{E}.
\]

Hence we finally obtain that our Hamiltonian has the form:

\[
H^{(ζ)}(\bar{λ}) = -n \sum \frac{λ_j^2}{2} + \sum_{i \neq j} \log |λ_i - λ_j| + \left( \frac{2}{β} - 1 \right) \sum \log ζ'(λ_j) + R(\bar{λ}) + n^2 Δ\mathcal{E}
\]
Linearization of $R(\tilde{\lambda})$

Consider the Hamiltonian

$$H_n(\tilde{\lambda}) = H_n^*(\tilde{\lambda}) + (1 - \frac{2}{\beta}) \sum \log \zeta'(\lambda_j) + \frac{1}{2} \sum_{k=1}^{M} \eta_k \left( \sum_j (\psi(\lambda_j) - (\psi_k, \rho_{sc})) \right)^2,$$

where $H_n^*$ is the Hamiltonian corresponding to $V^*(\lambda) = \lambda^2/2$. Write for any $1 \leq k \leq M$

$$\exp \left\{ \frac{\beta}{2} \eta_k \left( \sum_j (\psi(\lambda_j) - (\psi_k, \rho_{sc})) \right)^2 \right\}$$

$$= \sqrt{\frac{\beta}{8\pi}} \int \exp \left\{ \frac{\beta}{2} \left( \sqrt{\eta_k} \left( \sum_j (\psi_k(\lambda_j) - (\psi_k, \rho_{sc})) \right) u_k - u_k^2/4 \right) \right\}$$

and denote

$$h_{\tilde{u}}(\lambda) = \sum_{k=1}^{M} \sqrt{\eta_k} \psi_k(\lambda) u_k + \left( \frac{2}{\beta} - 1 \right) \log \zeta'(\lambda), \quad \dot{h}_{\tilde{u}} = h_{\tilde{u}} - (h_{\tilde{u}}, \rho_{sc}).$$
Global regime

We obtain

$$
\frac{Z_n[V, \beta]}{Z_n[V^*, \beta]} = \exp\left\{ \beta n^2 \Delta \mathcal{E} / 2 + n\left(1 - \frac{\beta}{2}\right)(\log \zeta', \rho_{sc}) \right\} \\
\cdot \left( \frac{\beta}{8\pi} \right)^{M/2} \int e^{-\beta(\bar{u}, \bar{u})/8} \langle e^{\beta N_n[\dot{h}_u]}/2 \rangle_{*, n} d\bar{u}
$$

Then for $\langle e^{\beta N_n[\dot{h}_u]}/2 \rangle_{*, n}$ the Johansson theorem yields

$$
Z_n[V, \beta] = Z_n[V^*, \beta] \exp \left\{ \frac{\beta}{2} n^2 \Delta \mathcal{E} + n\left(1 - \frac{\beta}{2}\right)(\log \zeta', \rho_{sc}) \right\} \\
\cdot \left( \frac{\beta}{8\pi} \right)^{M/2} \int \exp\left\{-\frac{\beta}{8}(\bar{u}, \bar{u}) + \frac{\beta}{8}(\overline{D_\sigma h_{\bar{u}}}, h_{\bar{u}})\right\}(1 + o(1)O((u, u)^2))d\bar{u}.
$$

The only fact which we need to prove is that the integral with respect to $\bar{u}$ is convergent.
Local bulk regime.

To study the gap probabilities we consider $\psi_{\Delta,\bar{m}}(\bar{\lambda}; \lambda_0)$.

After the change of variables we obtain that $\psi_{\Delta,\bar{m}}(\bar{\lambda}; \lambda_0)$ will be transform into the indicator function $\psi^{(\zeta)}_{\Delta,\bar{m}}(\bar{\lambda}; \zeta^{-1}(\lambda_0))$ of the same type but for the new system of intervals: each interval $\Delta_j = (a_j, b_j), j = 1, \ldots, k$ or

$$\lambda_0 + a_j/n\rho(\lambda_0) \leq \lambda \leq \lambda_0 + b_j/n\rho(\lambda_0)$$

should be replaced by

$$\lambda_0 + a_j/n\rho(\lambda_0) \leq \zeta(\lambda) \leq \lambda_0 + b_j/n\rho(\lambda_0)$$

$$\Leftrightarrow \zeta^{-1}(\lambda_0 + a_j/n\rho(\lambda_0)) \leq \lambda \leq \zeta^{-1}(\lambda_0 + b_j/n\rho(\lambda_0)).$$

But, e.g., for the left edge point we have

$$\zeta^{-1}(\lambda_0 + a_j/n\rho(\lambda_0)) = \zeta^{-1}(\lambda_0) + a_j/n\rho(\lambda_0) \zeta'(\lambda_0) + O(n^{-2})$$

$$= \zeta^{-1}(\lambda_0) + a_j/n\rho_{sc}(\zeta^{-1}(\lambda_0)) + O(n^{-2})$$

Hence we indeed have the indicator function of the same type.
To prove the universality of correlation functions in the weak form, it suffices to take arbitrary smooth functions $\phi_j(x)$ ($j = 1, \ldots, k$) and to consider the limits of the expectations of the functions of the form

$$
\Phi_k(\bar{\lambda}; \lambda_0) = \prod_{j=1}^{k} \left( n^{-1} \sum_{i=1}^{n} n \phi_j(n \rho(\lambda_0)(\lambda_i - \lambda_0)) \right), \quad \lambda_0 \in (-2 + \varepsilon, 2 - \varepsilon),
$$

we need to replace $\Phi_k(\bar{\lambda}; \lambda_0)$ by

$$
\Phi_k^{(\zeta)} = \prod_{j=1}^{k} \left( \sum_{i} \varphi_j(n \rho(\lambda_0)(\zeta(\lambda_i) - \lambda_0)) \right),
$$
Then in the case of the indicator functions we get

\[
\langle \Psi_{\Delta, \tilde{m}} \rangle_{V,n} = \langle \Psi^{(\zeta)}_{\Delta, \tilde{m}} \rangle_{H(\zeta)}
\]

\[
= I_n^{-1} \left( \frac{\beta}{8\pi} \right)^{M/2} \int e^{-\beta(\tilde{u}, \tilde{u})/8} d\tilde{u} \langle \Psi^{(\zeta)}_{\Delta, \tilde{m}} e^{\beta N_n [\hat{h}_u]/2} \rangle_{*, n} = \langle \Psi^{(\zeta)}_{\Delta, \tilde{m}} \rangle_{*, n}
\]

\[
+ I_n^{-1} \left( \frac{\beta}{8\pi} \right)^{M/2} \int e^{-\beta(\tilde{u}, \tilde{u})/8} d\tilde{u} \left( \langle \Psi^{(\zeta)}_{\Delta, \tilde{m}} e^{\beta N_n [\hat{h}_u]/2} \rangle_{*, n} - \langle \Psi^{(\zeta)}_{\Delta, \tilde{m}} \rangle_{*, n} \langle e^{\beta N_n [\hat{h}_u]/2} \rangle_{*, n} \right)
\]

where \( I_n \) is the normalization constant. It is \( e^{O(1)} \), so can give only some additional constant in the bounds.
The first step: real $h$

**Lemma**

Let $V_h = V^* + \frac{1}{n} h$ with real analytic $h$ such that $\|h^{(3)}\|_2 \leq C \log n$. Then

$$\left| \langle \Psi^{(\zeta)}_{\Delta, \bar{m}} e^{\beta N_n[\bar{h}]/2} \rangle_{\ast, n} - \langle \Psi^{(\zeta)}_{\Delta, \bar{m}} \rangle_{\ast, n} \right| \leq \varepsilon_n \to 0$$

Apply the change of variables procedure to $V_h = V^* + \frac{1}{n} h$. But then $h$ should be a "good" perturbation, i.e. the equilibrium density, corresponding to $V_h$, should have the support $[-2, 2]$. Hence, one should find $a, b$ such that the function

$$\tilde{h}(\lambda) = h(\lambda) - \ell(\lambda), \quad \ell(\lambda) := a\lambda^2 - b\lambda,$$

is a "good" perturbation ($a = (h', f_a), b = (h', f_b)$ with some fixed $f_a, f_b$), and apply the change of variables to $V_{\tilde{h}}$.

Since

$$\zeta_h(\lambda) = \lambda + n^{-1} \tilde{\zeta}_h(\lambda),$$

the corresponding integral operator kernel will be

$$\log \left(1 + \frac{1}{n} \frac{\zeta_h(\lambda) - \zeta_h(\mu)}{\lambda - \mu} \right) = \frac{1}{n} \tilde{\mathcal{L}}_h(\lambda, \mu)$$
Then, completing the change of variables, we obtain that it suffices to check that

\[
\langle \Psi^{(\zeta)}_{\Delta,\tilde{m}} (e^{\beta R(\bar{\lambda})/2n} - 1) \rangle_{*,n} \to 0
\]

It is easy, since

\[
\langle (e^{\beta R(\bar{\lambda})/2n} - 1)^2 \rangle_{*,n} \to 0.
\]

To remove \( \ell \) we use the following result

**Corollary from the result of Valko and Virag (09)**

\[
|\langle \Psi_{\Delta,\tilde{m}}(\bar{\lambda}, \lambda_0 + t/n) \rangle_{*,n} - \langle \Psi_{\Delta,\tilde{m}}(\bar{\lambda}, 0) \rangle_{*,n}| \leq \varepsilon_n \to 0, \quad n \to \infty,
\]

where the first bound is uniform for \( \lambda_0 \in [-2 + \varepsilon, 2 - \varepsilon] \), and the second relation is uniform in the same \( \lambda_0 \) and \( |t| \leq n^{1-\delta} \) if \( \delta > 0 \) is fixed.

Since there is no result similar to the above for the convergence of correlation functions in the case of \( \Phi_k(\bar{\lambda}, \lambda_0) \) we obtain that it coincides in the limit with \( \Phi_k(\bar{\lambda}, \lambda_0 + t(h)/n) \) where \( t(h) = (h', f) \) with some smooth \( f \) depending on \( V \).
The second step: complex h

Lemma

Let the analytic in $t \in \mathcal{D} = \{t : |t| \leq \log^{1/2} \varepsilon_n^{-1}, \Im t \geq 0\}$ functions $F_n$ satisfy two bounds:

$$|F_n(t)| \leq C_1 \varepsilon_n e^{t^2/2}, \quad -\log^{1/2} \varepsilon_n^{-1} \leq t \leq \log^{1/2} \varepsilon_n^{-1}, \quad \varepsilon_n < 1,$$

$$|F_n(t)| \leq C_2 e^{(\Re t)^2/2}, \quad t \in \mathcal{D}.$$

Then the inequality

$$|F_n(t)| \leq C \varepsilon_n^{1/2} |e^{t^2/2}|$$

holds for $t \in \mathcal{D}' := \frac{1}{6} \mathcal{D}$ with $C = C_1^{3/4} C_2^{1/4}$. 
Main results for the one-cut case

**Theorem 1 [S:13]**

Let $V$ be a smooth (possessing 7 derivatives) one-cut potential with $\sigma = [-2, 2]$ of generic behavior, and $\lambda_0 \in [-2 + \varepsilon, 2 - \varepsilon]$ with any fixed $\varepsilon > 0$. Then the following relations hold uniformly in $\lambda_0 \in [-2 + \varepsilon, 2 - \varepsilon]$:

(i) for any fixed nonintersecting intervals $\bar{\Delta}$, any fixed $\bar{m} \in \mathbb{N}^k$

$$\lim_{n \to \infty} \langle \psi_{\bar{\Delta}, \bar{m}}(\bar{\lambda}, \lambda_0) \rangle_{V,n} = \lim_{n \to \infty} \langle \psi_{\bar{\Delta}, \bar{m}}(\bar{\lambda}, 0) \rangle_{*,n}$$

(ii) any $\psi_\phi(\bar{\lambda}, \lambda_0)$ with compactly supported piece-wise continuous $\phi$

$$\lim_{n \to \infty} \langle \psi_\phi(\bar{\lambda}, \lambda_0) \rangle_{V,n} = \lim_{n \to \infty} \langle \psi_\phi(\bar{\lambda}, 0) \rangle_{*,n}.$$

(iii) There exists $s_* > 0$ depending on $V$, $\beta$ and $\lambda_0$ such that for any $k \geq 1$ and any $\Phi_k(\bar{\lambda}, \lambda_0)$ with compactly supported smooth (belonging to $C_1$) $\{\phi_j\}_{j=1}^k$ we have

$$\lim_{n \to \infty} \left| \langle \Phi_k(\bar{\lambda}, \lambda_0) \rangle_{V,n} - \sqrt{\frac{s_*}{2\pi}} \int dt e^{-s_* t^2/2} \langle \Phi_k(\bar{\lambda} + n^{-1} t, \zeta^{-1}(\lambda_0)) \rangle_{*,n} \right| = 0.$$
Multi-cut potentials and local edge regime

**Theorem 2 [S:13]**

Let $V$ be a real analytic multi-cut potential with $\sigma = \bigcup_{\alpha=1}^{q} \sigma_{\alpha}$ ($\sigma_{\alpha} = [a_{\alpha}, b_{\alpha}]$) of generic behavior. Then, for any $\lambda_0 \in \bigcup_{\alpha=1}^{q} [-a_{\alpha} + \varepsilon, b_{\alpha} - \varepsilon]$ the assertions (i)–(iii) holds.

For the local edge regime the procedure is the same, but one should consider the function $\tilde{\Psi}_{\Delta, \bar{m}}(\bar{\lambda}, b_{\alpha})$ which is the indicator of the set, where

$$N_{n}(b_{\alpha} + \Delta_{1}/n^{2/3}\gamma) = m_{1}, \ldots N_{n}(b_{\alpha} + \Delta_{k}/n^{2/3}\gamma) = m_{k}$$

**Theorem 3 [S:14]**

Let $V$ be a real analytic multi-cut potential with $\sigma = \bigcup_{\alpha=1}^{q} \sigma_{\alpha}$ ($\sigma_{\alpha} = [a_{\alpha}, b_{\alpha}]$)

$$\lim_{n \to \infty} \langle \tilde{\Psi}_{\Delta, \bar{m}}(\bar{\lambda}, b_{\alpha}) \rangle_{V,n} = \lim_{n \to \infty} \langle \tilde{\Psi}_{\Delta, \bar{m}}(\bar{\lambda}, 2) \rangle^{*,n}$$
Previous results

These results should be compared with

**Theorem [Bourgade, Erdos, Yau: 11-13]**

If $V$ is a one-cut potential of generic behavior and $|V^{(4)}| \leq C$, then for any $k$, and any smooth $\varphi_j$ with a compact support

$$
\lim_{n \to \infty} (2n^{\alpha-1})^{-1} \int_{-n^{-1+\alpha}}^{n^{-1+\alpha}} dt \left( \langle \Phi_k(\lambda_0 + t) \rangle_{V,n} - \langle \Phi_k(\lambda_0 + t) \rangle*,n \right) = 0
$$

Similar results were obtained for the edge universality.
Problems

(1) potentials with "hard edges"

(2) potentials with non generic behavior of the equilibrium density ("double scaling" case, etc.)