The many forms of rigidity for symplectic embeddings

based on work of Dan Cristofaro-Gardiner, Michael Entov, David Frenkel, Janko Latschev, Dusa McDuff, Dorothee Müller, FS, Misha Verbitsky
The problems

1. $E(1, a) \mapsto Z^4(A)$

2. $E(1, a) \mapsto C^4(A)$

3. $E(1, a) \mapsto P(A, bA), \quad b \in \mathbb{N}$

4. $E(1, a) \mapsto T^4(A)$
The problems

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2. \( E(1, a) \xrightarrow{s} C^4(A) \)
3. \( E(1, a) \xrightarrow{s} P(A, bA), \quad b \in \mathbb{N} \)
4. \( E(1, a) \xrightarrow{s} T^4(A) \)

where:

\[
E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} < 1 \right\}
\]

can assume: \( b = 1, \ a \geq 1 \)
and:

\[ Z^4(A) = D(A) \times \mathbb{C} \]
\[ C^4(A) = P(A, A) \]
\[ P(A, bA) = D(A) \times D(bA) \]
\[ T^4(A) = T^2(A) \times T^2(A) \]

moment map images (under \((z_1, z_2) \mapsto (\pi |z_1|^2, \pi |z_2|^2)) : \]
The answers
The answers

1. Gromov 1985: $E(1, a) \hookrightarrow Z^4(A)$ iff $A \geq 1$

Hence

$$c_{EZ}(a) := \inf \left\{ A \mid E(1, a) \hookrightarrow Z^4(A) \right\} \equiv 1$$

TOTAL symplectic rigidity, NO structure
To get "some structure", truncate!
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First: Truncate all down to $C^4(A)$:

$$c_1(a) := \inf \left\{ A \mid E(1, a) \hookrightarrow C^4(A) \right\} \geq \sqrt{\frac{a}{2}} \quad \text{(Volume constraint)}$$
2. Frenkel–Müller 2014, based on McDuff–S 2012:

c₁(a) starts with the **Pell stairs**:

**Pell numbers:** \( P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2} \)

**HC Pell numbers:** \( H_0 = 1, H_1 = 1, H_n = 2H_{n-1} + H_{n-2} \)
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**HC Pell numbers:** \( H_0 = 1, \) \( H_1 = 1, \) \( H_n = 2H_{n-1} + H_{n-2} \)

Form the sequence

\[
\left( \gamma_1, \gamma_2, \gamma_3, \ldots \right) := \left( \frac{P_1}{H_0}, \frac{H_2}{2P_1}, \frac{P_3}{H_2}, \frac{H_4}{2P_3}, \ldots \right)
\]

\[
= \left( 1, \frac{3}{2}, \frac{5}{3}, \frac{17}{10}, \ldots \right)
\]

\[
\rightarrow \frac{\sigma}{\sqrt{2}} \ \text{where} \ \sigma = \sqrt{2} + 1 \ \text{the silver ratio}
\]
An application:

Biran 1996:

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<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>$\geq 8$</th>
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<tbody>
<tr>
<td>$k$</td>
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<tr>
<td>$p_k$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{8}{9}$</td>
<td>$\frac{9}{10}$</td>
<td>$\frac{48}{49}$</td>
<td>$\frac{224}{225}$</td>
<td>1</td>
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where $p_k =$ percentage of volume of $[0, 1]^4 \subset \mathbb{R}^4$ that can be symplectically filled by $k$ disjoint equal balls

What are these numbers?
The function $c_1$ explains Biran’s list:

$$d_k := \inf \left\{ A \left| \bigsqcup_k B^4(1) \xrightarrow{s} C^4(A) \right. \right\}$$

Since $p_k = \frac{k \cdot \frac{1}{2}}{d_k^2}$, his list becomes

<table>
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<tr>
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<td>1</td>
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<td>$\frac{5}{3}$</td>
<td>$\frac{7}{4}$</td>
<td>$\frac{15}{8}$</td>
<td>$\sqrt{\frac{k}{2}}$</td>
</tr>
</tbody>
</table>

and (McDuff 2009): $d_k = c_1(k)$, ie

$$\bigsqcup_k B^4(1) \xrightarrow{s} C^4(A) \iff E(1, k) \xrightarrow{s} C^4(A)$$
Hence: It was worthwhile to elongate the domain:

\[ B^4(1) \leadsto E(1, a) \]

Now: Also elongate the target:

\[ C^4(A) = P(A, A) \leadsto P(A, bA), \quad b \geq 1 \]
Hence: It was worthwhile to elongate the domain:

\[ B^4(1) \sim E(1, a) \]

Now: Also elongate the target:

\[ C^4(A) = P(A, A) \sim P(A, bA), \quad b \geq 1 \]

For \( a, b \geq 1 \)

\[ c_b(a) = \inf \left\{ A \mid E(1, a) \xrightarrow{s} P(A, bA) \right\} \]

1-parametric family of problems

Volume constraint: \( c_b(a) \geq \sqrt{\frac{a}{2b}} \)
3. Cristofaro–Gardiner, Frenkel, S 2016 \(c_b(a)\) is given by ...
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So: fine structure of symplectic rigidity in \( c_1 \)

first disappears (as \( b \to 2 \)), then reappears (as \( b \to \infty \))
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So: fine structure of symplectic rigidity in $c_1$
first disappears (as $b \to 2$), then reappears (as $b \to \infty$)

Remarks

1. “The same” holds true for $b \in \mathbb{R}_{\geq 2}$ (no proof)
2. The first two linear steps, and the affine step of the Pell stairs are stable, the other steps disappear

Open problem: How does the Pell stairs disappear?

Understand $c_{1+\varepsilon}$
4. $E(1, a) \xrightarrow{s} T^4(A)$:

Note: $C^4(A)$ compactifies to both $S^2(A) \times S^2(A)$ and $T^4(A)$

Fact: $c_1(a) = \inf \left\{ A \mid E(1, a) \xrightarrow{s} C^4(A) \right\}$

$= \inf \left\{ A \mid E(1, a) \xrightarrow{s} S^2(A) \times S^2(A) \right\}$
4. $E(1, a) \hookrightarrow \mathbb{T}^4(A)$:

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Fact: $c_1(a) = \inf \left\{ A \mid E(1, a) \hookrightarrow C^4(A) \right\}$

$= \inf \left\{ A \mid E(1, a) \hookrightarrow S^2(A) \times S^2(A) \right\}$

On the other hand (Latschev–McDuff–S, Entov–Verbitsky 2014):

$E(1, a) \hookrightarrow \mathbb{T}^4(A)$ whenever $\text{Vol } E(1, a) < \text{Vol } \mathbb{T}^4(A)$

("Total flexibility")
But:

- It is unknown whether \( \text{Emb}(E(1, a), T^4(A)) \) is connected (hence \(<\))
- Hidden rigidity (Biran): Assume that

\[
\varphi: B^4(a) \leftrightarrow T^4(1), \quad \text{Vol } B^4(a) = \frac{3}{4},
\]
\[
\psi: \coprod_2 B^4(b) \leftrightarrow T^4(A), \quad \text{Vol } \coprod_2 B^4(b) = \frac{2}{3}.
\]

Then \( \text{Im } \psi \subset \text{Im } \varphi \) is impossible (by Gromov’s 2-ball theorem)
Ideas of the proof

Common ingredient: from $B_4(a) \hookrightarrow (M_4, \omega)$ get symplectic form $\omega$ on the blow-up $\pi: M_1 \to M$ in class $\pi^*[\omega] - a e = PD(E)$.

Conversely: If $\pi^*[\omega] - a e$ has a symplectic representative, non-degenerate along $\Sigma$, then can "blow-down", get $B_4(a) \hookrightarrow (M, \omega)$.
Ideas of the proof

**Common ingredient:**

from $\overline{B}^4(a) \xrightarrow{s} (M^4, \omega)$ get symplectic form $\omega_a$ on the **blow-up**

$$\pi: M_1 \to M$$

in class $\pi^*[\omega] - a\, e$

$e = \text{PD}(E), \ E = [\Sigma]$
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Conversely: If $\pi^*[\omega] - a\, e$ has a symplectic representative, non-degenerate along $\Sigma$,

then can “blow-down”, get $\mathbb{B}^4(a) \xleftarrow{s} (M, \omega)$
\[ E(1, 2) \hookrightarrow C^4(1 + \varepsilon) \quad \text{(easy!)} \]
\( E(1, 2) \hookrightarrow C^4(1 + \varepsilon) \quad \left( \coprod_2 B^4(1) \hookrightarrow C^4(1 + \varepsilon) \text{ easy!} \right) \)

omit \( \varepsilon, \delta \) ...

compactify \( C^4(1) = D(1) \times D(1) \) to \( M = S^2(1) \times S^2(1) \)

\( \Delta := \{(z, z)\} \subset S^2 \times S^2 : \text{holomorphic} \)
\[ E(1, 2) \xrightarrow{s} C^4(1 + \varepsilon) \quad \left( \coprod_2 B^4(1) \xrightarrow{s} C^4(1 + \varepsilon) \text{ easy!} \right) \]

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\(\Delta := \{(z, z)\} \subset S^2 \times S^2 : \text{holomorphic}\)

blow-up \(M\) twice “at the right points” by size \(\lambda = \frac{1}{3}\)

get holomorphic sphere \(\Delta_2 \subset M_2\) in class

\[ [\Delta_2] = S_1 + S_2 - E_1 - E_2 \]

and a chain of spheres \(C_1 \cup C_2\) “bounding” \(E(\lambda, 2\lambda)\)

and a symplectic form \(\omega_\lambda\) in class

\[ [\omega_\lambda] = s_1 + s_2 - \lambda(e_1 + e_2) \]
Wish to "inflate" the form

$$\omega_\lambda \text{ in } [\omega_\lambda] = s_1 + s_2 - \lambda(e_1 + e_2)$$

to

$$\omega_1 \text{ in } [\omega_1] = s_1 + s_2 - (e_1 + e_2)$$

Inflation along $e_1 + e_2$ is impossible, but can inflate along

$$\text{PD}(\Delta_2) = s_1 + s_2 - e_1 - e_2 :$$
Wish to “inflate” the form

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Inflation along \( e_1 + e_2 \) is impossible, but can inflate along

\[ \text{PD}(\Delta_2) = s_1 + s_2 - e_1 - e_2 : \]

\( \forall \tau > 0 \) there exists a symplectic form \( \Omega_\tau \) in class

\[ [\omega_\lambda] + \tau \text{PD}[\Delta_2] = (1 + \tau)(s_1 + s_2) - (\lambda + \tau)(e_1 + e_2) \]

Get

\[ E(\lambda + \tau, 2(\lambda + \tau)) \overset{\mathcal{S}}\hookrightarrow \mathbb{C}^4(1 + \tau) \]

Hence

\[ E(1, 2) \overset{\mathcal{S}}\hookrightarrow \mathbb{C}^4\left(\frac{1+\tau}{\lambda+\tau}\right) \]
E(1, 2) ↦ T^4(1)

Cannot do inflation, since there are no $J$-curves to inflate along

But: Use existence of Kähler forms on blow-ups

(much stronger than Nakai–Moishezon in the algebraic case)
E(1, 2) ↪ T^4(1)

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But: Use existence of Kähler forms on blow-ups (much stronger than Nakai–Moishezon in the algebraic case).

$(M, J)$ Kähler.

$H^1_{J, 1}(M; \mathbb{R})$: classes represented by $J$-invariant closed 2-forms.

Candidates for Kähler classes: $C^1_{+, 1}(M, J) := \left\{ \alpha \in H^1_{J, 1}(M; \mathbb{R}) \mid \alpha^m([V]) > 0 \ \forall \ \text{complex subv. } V^m \subset (M, J) \right\}$
Demainly–Paun 2004 The Kähler cone of \((M, J)\) is one of the connected components of \(C_{+}^{1,1}(M, J)\)
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Hence: $E(1, 2) \hookrightarrow T^4(1, 1 + \varepsilon)$ (for $\varepsilon$ irrational)

since there is a Kähler $J$ without $J$-curves

(positivity on exceptional divisors and on $M_2$ clear)
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For \(T^4(1, 1)\) use approximation of \((T^4, \omega, J)\) by \(J\) close to \(J\) without \(J\)-curves
Kähler for some \(\omega_J\) with \([\omega_J]\) close to \([\omega]\) (Kodaira–Spencer)
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“Yesterday”: Entov–Verbitsky proved flexibility of ellipsoid packings of Kähler tori in all dimensions
by directly blowing-up \(E(a_1, \ldots, a_n)\) (\(a_j\) relatively prime) and resolving the cyclic singularities (Hironaka)
Can assume $a \in \mathbb{Q}$

weight sequence

$$w(a) = (1, \ldots, 1, w_1^{\times \ell_1}, \ldots, w_N^{\times \ell_N})$$
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$$w(a) = (1, \ldots, 1, w_1^{\times \ell_1}, \ldots, w_N^{\times \ell_N})$$

Examples:

$$w(3) = (1, 1, 1), \quad w\left(\frac{5}{3}\right) = \left(1, \frac{2}{3}, \left(\frac{1}{3}\right)^{\times 2}\right)$$
$B(\mathbf{w}(a)) := \bigsqcup_i B(w_i)$ (with multiplicities)
\[ B(w(a)) := \bigsqcup_i B(w_i) \text{ (with multiplicities)} \]

McDuff, Frenkel–Müller:

\[ E(1, a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b) \iff B(w(a)) \bigsqcup B(\lambda) \bigsqcup B(\lambda b) \overset{s}{\hookrightarrow} B(\lambda(b + 1)) \]
Use three methods!

Method 1: obstructive classes

Method 2: dual version: reduction “at a point”

Method 3: ECH capacities
Method 1: McDuff, Polterovich, Biran, Li–Lu:

\[ E(1, a) \overset{s}{\leftrightarrow} P(\lambda, \lambda b) \iff \]

(i) \[ \lambda \geq \sqrt{\frac{a}{2b}} \quad \text{volume constraint} \]
Method 1: McDuff, Polterovich, Biran, Li–Lu:

\[ E(1, a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b) \iff \]

(i) \( \lambda \geq \sqrt{\frac{a}{2b}} \) volume constraint

(ii) \( \lambda \geq \frac{\langle m, w(a) \rangle}{d + be} \) for all solutions \((d, e; m) \in \mathbb{N}^3\) of

\[ \sum_i m_i = 2(d + e) - 1, \quad \sum_i m_i^2 = 2de + 1 \]

constraint from \(J\)-spheres
For $b \geq 2$ all obstructions come from

$$E_n = (n, 1; 1^{(2n+1)}) \quad \text{(linear steps)}$$

$$F_n = (n(n+1), n+1; n+1, n^{(2n+3)}) \quad \text{(affine step)}$$
Method 3:

Hutchings, McDuff:

**ECH capacities are complete invariants** for our problem:

\[ E(a, b) \xrightarrow{s} E(c, d) \iff c_k(E(a, b)) \leq c_k(E(c, d)) \text{ for all } k \]
Method 3:

Hutchings, McDuff:

**ECH capacities are complete invariants** for our problem:

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In particular

\[ E(1, a) \overset{s}{\rightarrow} P(\lambda, \lambda b) \iff c_k(E(1, a)) \leq \lambda c_k(E(1, 2b)) \text{ for all } k \]
Method 2: reduction at a point

\[(\mu; a_1, \ldots, a_k) \in \mathbb{R}^{1+k} \text{ ordered if } a_1 \geq \cdots \geq a_k\]
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defect of an ordered vector \((\mu; \mathbf{a})\): \[\delta = \mu - (a_1 + a_2 + a_3)\]
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defect of an ordered vector \((\mu; a)\): \( \delta = \mu - (a_1 + a_2 + a_3) \)

\((\mu; a)\) ordered is reduced if \( \delta \geq 0 \) and \( a_i \geq 0 \)
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$$(\mu; a_1, \ldots, a_k) \in \mathbb{R}^{1+k} \text{ ordered if } a_1 \geq \cdots \geq a_k$$

defect of an ordered vector $(\mu; a)$: $$\delta = \mu - (a_1 + a_2 + a_3)$$

$(\mu; a)$ ordered is reduced if $\delta \geq 0$ and $a_i \geq 0$

Cremona transform $\text{Cr}: \mathbb{R}^{1+k} \to \mathbb{R}^{1+k}$

$$ \begin{aligned} (\mu; a) \mapsto (\mu+\delta; a_1+\delta, a_2+\delta, a_3+\delta, a_4, \ldots, a_k) \end{aligned}$$
Method 2: reduction at a point

\((\mu; a_1, \ldots, a_k) \in \mathbb{R}^{1+k} \) ordered if \(a_1 \geq \cdots \geq a_k\)

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\((\mu; a) \mapsto (\mu + \delta; a_1 + \delta, a_2 + \delta, a_3 + \delta, a_4, \ldots, a_k)\)

Cremona move: reorder \(\circ \text{Cr}\)
Tian-Jun Li, Buse–Pinsonnault:

\[ E(1, a) \mathrel{s\hookrightarrow} P(\lambda, \lambda b) \quad \iff \quad \left( \lambda(b + 1); \lambda b, \lambda, \mathbf{w}(a) \right) \text{ reduces under finitely many Cremona moves to a reduced vector} \]
Tian-Jun Li, Buse–Pinsonnault:

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Dynamical interpretation:
\[ X_k = \text{blow up of } \mathbb{CP}^2 \text{ in } k \text{ points} \]
$X_k = \text{blow up of } \mathbb{C}P^2 \text{ in } k \text{ points}$

$\mathcal{E}(X_k) = \text{exceptional classes } E \in H_2(X_k; \mathbb{Z}) \text{ with } c_1(E) = 1 \text{ and } E \cdot E = -1 \text{ that can be represented by smoothly embedded spheres}$
$X_k =$ blow up of $\mathbb{CP}^2$ in $k$ points

$\mathcal{E}(X_k) =$ exceptional classes $E \in H_2(X_k; \mathbb{Z})$ with $c_1(E) = 1$ and $E \cdot E = -1$ that can be represented by smoothly embedded spheres

$\mathcal{C}(X_k) =$ symplectic cone:

$$\left\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \text{ for all } E \in \mathcal{E}(X_k) \right\}$$
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$\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \text{ for all } E \in \mathcal{E}(X_k) \}$

$\overline{\mathcal{P}}_+^k = \text{positive cone:}$

$\{(\mu; a) \in \mathbb{R}^{1+k} \mid \mu, a_1, \ldots, a_k \geq 0, \|a\| \leq \mu\}$
$X_k = \text{blow up of } \mathbb{C}P^2 \text{ in } k \text{ points}$

$\mathcal{E}(X_k) = \text{exceptional classes } E \in H_2(X_k; \mathbb{Z}) \text{ with } c_1(E) = 1 \text{ and } E \cdot E = -1 \text{ that can be represented by smoothly embedded spheres}$

$\mathcal{C}(X_k) = \text{symplectic cone:}$

$$\left\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \text{ for all } E \in \mathcal{E}(X_k) \right\}$$

$\overline{\mathcal{P}^k_+} = \text{positive cone:}$

$$\left\{ (\mu; \mathbf{a}) \in \mathbb{R}^{1+k} \mid \mu, a_1, \ldots, a_k \geq 0, \|\mathbf{a}\| \leq \mu \right\}$$

$\mathcal{R} = \text{reduced vectors}$

Then $\mathcal{R} \subset \overline{\mathcal{C}(X_k)} \subset \overline{\mathcal{P}^k_+}$