Calabi–Yau mirror symmetry
from categories to curve-counts

Tim Perutz (University of Texas at Austin)
Joint work with Nick Sheridan (Princeton/IAS)

November 15, 2013
Mirror symmetry relates the \textit{A-model} geometry of one (suitably enhanced) Calabi–Yau manifold $X$ to the \textit{B-model} geometry of another, $\check{X}$.

- \textbf{(A)} symplectic
  - Fukaya categories ("open string" invariants)
  - Quantum cohomology,
  - counting holomorphic curves ("closed string" invariants)

- \textbf{(B)} algebro-geometric
  - derived categories of coh. sheaves ("open string invariants")
  - the Gauss–Manin connection in de Rham cohomology
  - periods of a volume-form; ("closed string invariants")

Mirror symmetry is involutory: $\check{X} = X$. In my presentation, A and B will appear asymmetric, because I’m only showing you part of the data.
Gist of the lecture

- Mirror CY pairs
  \( X \): symplectic manifold with \( c_1(TX) = 0 \);
  \( \mathcal{X} \): smooth projective variety with trivialized canonical bundle.

- Open string (homological) mirror symmetry claims:
  
  derived Fukaya category \( \mathcal{F}(X) \) is equivalent to 
  derived category of coherent sheaves \( D^b \text{Coh}(\mathcal{X}) \)

- We’ll assume
  
  subcat. \( A \) of \( \mathcal{F}(X) \) \( \simeq \) subcat. \( B \) of \( D^b \text{Coh}(\mathcal{X}) \)
  where \( B \) generates \( D^b \text{Coh}(\mathcal{X}) \).

- From this we’ll deduce homological mirror symmetry.
- We’ll also deduce parts of closed string mirror symmetry:

  \[ QH^\bullet(X) \cong H^*(\Lambda^* T\mathcal{X}); \]

  counts of rational curves in \( X \) = periods of volume form on \( \mathcal{X} \).
A-side: symplectic set-up

- An integral symplectic CY manifold consists of:
  - a compact manifold $X^{2n}$;
  - a symplectic form $\omega$ on $X$;
  - a codimension 2 symplectic submanifold $D \subset X$;
  - a 1-form $\theta \in \Omega^1(X \setminus D)$ such that $d\theta = \omega|_{X \setminus D}$; and
  - Nowhere-vanishing complex volume form $\Omega \in C^\infty(X; K_X)$.

- Main class of examples:
  - $X$ complex manifold;
  - $\omega$ curvature of a hermitian holomorphic line bundle;
  - $D = s^{-1}(0)$ where $s$ is a holomorphic section;
  - $\theta$ the connection in the trivialization $s$;
  - $\Omega$ a holomorphic volume form.

- Warning: we only expect to form a mirror to $X$ when it’s complex and on the brink of degenerating completely!
B-side: algebraic set-up

- Laurent series field $\mathbb{K} = \mathbb{C}[q^{-1}][q]$.
- Think of $\text{Spec} \mathbb{K}$ as an algebraic ‘punctured disc’: home for Laurent expansions of meromorphic functions on $\Delta^*(r)$.
- Our B-side CY varieties will be smooth $n$-dimensional projective varieties $\tilde{X}$ over $\mathbb{K}$, with ‘holomorphic volume forms’
  \[ \tilde{\Omega} \in H^0(\Lambda^n T^*\tilde{X}). \]
- Were $\tilde{X}$ defined by homogeneous polynomials whose coefficients (in $\mathbb{K}$) had positive radius of convergence, we could turn $\tilde{X}$ into a holomorphic family $\tilde{X} \to \Delta^*(r)$ over a punctured disc.
  E.g.
  \[
  \{ Y^2Z = X^3 + a(q)XZ^2 + b(q)Z^3 \} \subset \Delta^*(r) \times \mathbb{C}P^2.
  \]
  We do not want to assume convergence.
B-side: quasi-unipotent monodromy

- Algebraic de Rham cohomology

\[ H^\bullet_{\text{DR}}(\check{\mathcal{X}}/\mathbb{K}) = \check{H}^*(\Omega^*_\check{\mathcal{X}}/\text{Spec} \mathbb{K}) \quad (\text{graded } \mathbb{K}\text{-algebra}) \]

carries an automorphism \( T \in \text{Aut} H^\bullet_{\text{DR}}(\check{\mathcal{X}}/\mathbb{K}) \): the monodromy of the Gauss–Manin connection \( \nabla_{d/dq} \).

- If \( \check{\mathcal{X}} \) is the Laurent expansion of a holomorphic family with fibers \( \check{X}_q \), \( T \) can be identified with the monodromy around \( S^1(r) \) acting in \( H^*_\text{sing}(\check{X}_q; \mathbb{K}) \).

- **Quasi-unipotency**: after a substitution \( q \mapsto q^k \), the monodromy is unipotent of exponent \( n + 1 \):

\[ (T - I)^{n+1} = 0. \]

[Griffiths–Landman–Grothendieck; N. Katz]

T. Perutz  |  Calabi–Yau mirror symmetry from categories to curve-counts
Maximally unipotent monodromy

- We assume that the monodromy $T$ of $\tilde{X}$ is maximally unipotent:

  $$(T - I)^{n+1} = 0, \quad (T - I)^n \neq 0.$$  

- Maximal unipotency means that $\tilde{X}$ is a punctured disc around a point in the deepest stratum of CY moduli space.

- Example: a CY hypersurface in projective space degenerating to a union of hyperplanes.

- Maximal unipotency is a reasonable assumption: it is needed for $\tilde{X}$ to have a (closed-string) mirror $X$. 

T. Perutz  Calabi–Yau mirror symmetry  from categories to curve-counts
A-side: closed string invariants

- The small quantum cohomology $QH^*(X) = H^*(X; \mathbb{K})$ is a graded $\mathbb{K}$-algebra under the quantum product $\star$ (associative, graded, unital, graded-commutative).

- Structure constants of $\star$ ‘count’ pseudo-holomorphic spheres $u: S^2 \to X$ weighted as $q^{u \cdot D}$.

- Integration

$$\int_X : QH^{2n}(X) \to \mathbb{K}$$

makes $QH^*(X)$ into a Frobenius algebra. That is, $QH^i(X)$ is perfectly paired with $QH^{2n-i}(X)$ via

$$(a, b) \mapsto \int_X a \star b.$$
The tangential cohomology

\[ HT^*(\mathcal{X}) = \bigoplus_{p+q=*} H^p(\Lambda^q T\mathcal{X}) \]

is also graded \( \mathbb{K} \)-algebra.

The volume form \( \tilde{\Omega} \in H^0(\Lambda^n T^*\mathcal{X}) \) determines a trace map

\[ \text{tr}: HT^{2n}(\mathcal{X}) \to \mathbb{K} \]

making \( HT^*(\mathcal{X}) \) a Frobenius algebra.

In complex-analytic terms, represent \( \eta(q) \in H^n(\Lambda^n T\mathcal{X}_q) \) by \( \eta(q) \in C^\infty(\Lambda^{0,n} T^* \otimes \Lambda^n T) \). Contract \( \eta(q) \) with \( \tilde{\Omega}_q \) to get a \((0,n)\)-form \( \iota(\eta)\tilde{\Omega} \). Put

\[ \text{tr} \eta(q) = \int_{\tilde{\mathcal{X}}_q} \tilde{\Omega}_q \wedge \iota(\eta)\tilde{\Omega}_q. \]
Distinguished degree 2 classes

A-side

- symplectic class \( D \in QH^2(X) \).
- \( D^* = D^{-1} + O(q) \) is non-zero.

B-side

- Kodaira–Spencer class \( \theta \in H^1(T\tilde{X}) \subset HT^2(\tilde{X}) \) for the vector field \( q \frac{d}{dq} \) on the punctured disc.
  If \( \tilde{X} \) is the Laurent expansion of a map from a punctured disc to CY moduli space then \( \theta \) is the derivative of this map.
- Maximal degeneration assumption:
  \[
  \theta^n \neq 0 \in H^n(\Lambda^n T\tilde{X})
  \]

Complex analytic case: maximally unipotent monodromy \( \Rightarrow \) maximal degeneration. Proof uses mixed Hodge theory [Deligne; Schmid]
A-side: open string invariants

- The Fukaya category for $X$ relative to $D$ is a $\mathbb{K}$-linear $A_\infty$ category. Its objects are closed, exact Lagrangian submanifolds [with gradings and pin structures].

  \[ L^n \subset X^{2n} \setminus D : \quad \theta|_L = d(\text{some function on } L) \]

- Morphism space $\text{hom}(L_0, L_1)$ is Floer’s cochain space

  \[ \text{CF}(L_0, L_1) = \mathbb{K}^{L_0 \cap L_1} \]

  when $L_0 \pitchfork L_1$.

- $A_\infty$ operations $\mu^d$, $d \geq 0$, count pseudo-holomorphic $(d + 1)$-gons $u$ in $X$, bounded by exact Lagrangians, weighted by $q^{u \cdot D}$.

- Working only with exact Lagrangians in $X \setminus D$ results in a large saving in foundational complexity [Sheridan].
B-side: open string invariants

- Work with a DG model Perf $\hat{\mathcal{X}}$ for the derived category.
- A DG category is the same thing as an $A_{\infty}$-category with vanishing higher compositions $\mu^d$ ($d \geq 3$).
- **Objects**: finite complexes of algebraic vector bundles.
- **Morphism spaces**: Čech cochain complexes

$$\check{C}^\bullet(\mathcal{U}; \text{Hom}(\mathcal{E}, \mathcal{F}))$$

with respect to a fixed open affine cover $\mathcal{U}$.

- The cohomology of the hom-space is $\mathbb{R}\text{Hom}^\bullet(\mathcal{E}, \mathcal{F})$, the derived sheaf homomorphisms from $\mathcal{E}$ to $\mathcal{F}$.
Weak CY structures

- A weak CY\(_{n}\) structure on an \(A_\infty\)-category \(\mathcal{C}\) is a quasi-isomorphism of \((\mathcal{C}, \mathcal{C})\)-bimodules

\[
\beta : \mathcal{C} \to \mathcal{C}^\vee[n] \quad \text{such that} \quad \beta^\vee \simeq \beta.
\]

- Non-degenerate symmetric bilinear form on a category.

- \(\mathcal{F}(X, D)\) has an intrinsic weak CY\(_{n}\) structure:

\[
HF^*(L_0, L_1) \cong HF^{n-*}(L_1, L_0)^\vee.
\]

- Serre duality and the volume form \(\check{\Omega}\) determine a CY\(_{n}\) structure on Perf \(\check{X}\):

\[
\text{RHom}^*(\check{E}, \mathcal{F}) \cong \text{RHom}^{n-*}(\mathcal{F}, \check{E})^\vee.
\]
Homological mirror symmetry (HMS)

- Mirror symmetry identifies mirror pairs of CY varieties. Some are relatively simple (e.g. the mirror to the quintic 3-fold). Some are very sophisticated (e.g. Gross–Siebert program).
- A version of Kontsevich’s HMS conjecture predicts a quasi-equivalence

\[ \psi : F(X, D) \rightarrow \text{Perf } \tilde{X} \]

of \( \mathbb{K} \)-linear weak CY\( n \) \( A_\infty \)-categories.
- The underline denotes a certain algebraic enlargement of \( F(X, D) \) that I’m not going to explain.
Weak HMS

- We shall assume *weak homological mirror symmetry*. That is, we suppose given
  1. Some Lagrangians forming a full subcategory $\mathcal{A} \subset \mathcal{F}(X, D)$;
  2. Some perfect complexes a full subcategory $\mathcal{B} \subset \text{Perf } \check{X}$ which split-generates $\text{Perf } \check{X}$;
  3. an $A_\infty$-functor

\[ \psi: \mathcal{A} \to \mathcal{B} , \]

respecting weak CYn structures, such that

\[ H^*\psi: H^*\mathcal{A} \to H^*\mathcal{B} \]

is a categorical isomorphism.

- Split generation means that the closure of $\mathcal{B}$ is $\text{Perf } \check{X}$ under the following operations: shifts, mapping cones, isomorphisms, passing to direct summands.

- One could take $\mathcal{B} = \{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots \}$. 
Statement of results

\((X^{2n}, D)\): integral symplectic CY \(2n\)-manifold.
\((\check{X}, \check{\Omega})\): smooth, projective \(\mathbb{K}\)-variety, max. degenerate \((\theta^n \neq 0)\)
e.g. analytic family with maximally unipotent monodromy.

Theorem (Sheridan–P.)

Assume weak HMS, i.e., \(\psi: A \xrightarrow{\sim} B\), where \(B\) split-generates \(\text{Perf} \, \check{X}\). Then

1. \(A\) split-generates \(\mathcal{F}(X, D)\), and so full HMS holds.
2. \(\psi\) determines an isomorphism of graded \(\mathbb{K}\)-algebras

\[ \kappa: QH^*(X) \to HT^*(\check{X}) \]

preserving the distinguished degree 2 elements: \(\kappa(D) = \theta\).
3. \(\psi\) also preserves Frobenius traces, and consequently

\[ \int_X D^{*n} = \text{tr} \theta^n. \]
Caveat

We have not (yet) proved that the volume form $\check{\Omega}$ has the standard form demanded by Hodge-theoretic mirror symmetry. Thus the enumerative formula

$$\int_X D^{*n} = \text{tr} \theta^n = \int_{\check{\mathcal{X}}_q} \check{\Omega}_q \wedge \nu(\theta^n)\check{\Omega}_q$$

is not yet in practical form for counting curves.
Proof that $\mathcal{A}$ split-generates $\mathcal{F}(X, D)$: A-side

- The closed-open string map is a map of rings

$$CO|_{\mathcal{A}} : QH^*(X) \to HH^*(\mathcal{A})$$

whose target is Hochschild cohomology
($= $ natural transformations $\text{id}_{\mathcal{A}} \Rightarrow \text{id}_{\mathcal{A}}$).

- Abouzaid’s generation criterion for Fukaya categories dual version [Abouzaid–Fukaya–Oh–Ohta–Ono]:
  $CO|_{\mathcal{A}}$ injective in top degree $2n \Rightarrow \mathcal{A}$ split-generates $\mathcal{F}(X, D)$

- $QH^{2n}(X)$ is generated by $D^{*n}$, so we want $CO|_{\mathcal{A}}(D^{*n}) \neq 0$.

- Sheridan earlier observed that

$$CO(D) = \left[ q \frac{d\mu^*_{\mathcal{A}}}{dq} \right] \in HH^2(\mathcal{A}).$$
Proof that $\mathcal{A}$ split-generates $\mathcal{F}(X, D)$: B-side

- The twisted Hochschild–Kostant–Rosenberg map is a ring isomorphism

$$\text{HKR}: HT^*(X) \rightarrow HH^*(\text{Perf } \check{X}).$$

[... Calaque–van den Bergh–Rossi]

- $\text{HKR}(\theta)$ is the categorical analog of the Kodaira–Spencer class, describing how $\text{Perf } \check{X}$ varies in $q$.
- Since $\mathcal{B}$ split-generates $\text{Perf } \check{X}$ by assumption,

$$HH^*(\text{Perf } \check{X}) = HH^*(\mathcal{B}).$$
Given weak HMS, we have a composite ring map $\kappa$:

$$
\begin{array}{rcccl}
QH^*(X) & \xrightarrow{CO|_A} & HH^*(A) \\
\downarrow \kappa & & \downarrow HH^*(\psi) \\
HT^*(\tilde{X}) & \xleftarrow{\text{HKR}^{-1}} & HH^*(\text{Perf } \tilde{X}) & \xrightarrow{} & HH^*(B)
\end{array}
$$

Under these maps:

- $D \mapsto [q(d\mu/dq)] \mapsto \text{categorical KS} \mapsto \text{geometric KS}.$
- That is, $\kappa(D) = \theta$. So $\kappa(D^*) = \theta^* \neq 0$. Abouzaid's criterion tells us that $A$ split-generates.
Isomorphism of quantum and tangential cohomology

We now have

$$
\kappa : QH^*(X) \to HT^*(\tilde{X})
$$

ring map, $$\kappa(D^{*n}) \neq 0$$.

By Poincaré duality for $$QH^*(X)$$, $$CO|_A$$ is injective.

From this it follows that $$CO|_A$$ is surjective [Ganatra]. So $$CO|_A$$ is an isomorphism.

Hence $$\kappa$$ is a ring isomorphism.

Key new ingredients: use Abouzaid’s criterion; role of maximal degenerations; recent advances on HKR. Couple these with homological algebra.