Joint equidistribution of adelic torus orbits and families of twisted $L$-functions

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First Linnik problem

For \( D \in \mathbb{N} \) put \( \mathcal{R}_D = \{(x, y, z) \in \mathbb{Z}_\text{prim}^3 : x^2 + y^2 + z^2 = D\} \)

Legendre: \( \mathcal{R}_D \neq \emptyset \) iff \( D \in \mathbb{D} \), where \( \mathbb{D} = \{D \not\equiv 0, 4, 7 \mod 8\} \).

Gauss, Siegel, Dirichlet: for \( D \in \mathbb{D} \): \( |\mathcal{R}_D| = D^{1/2+o(1)} \).

Write

\[
\mathcal{I}_D = \left\{ \frac{v}{\|v\|} : v \in \mathcal{R}_D \right\} \subset S^2 = \{x^2 + y^2 + z^2 = 1\}.
\]

Let \( \mu_{S^2} \) be the normalized Lebesgue measure on the sphere \( S^2 \).
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Write

$$\mathcal{I}_D = \left\{ \frac{v}{\|v\|} : v \in R_D \right\} \subset S^2 = \{x^2 + y^2 + z^2 = 1\}.$$ 

Let $\mu_{S^2}$ be the normalized Lebesgue measure on the sphere $S^2$.

Conjecture A: Equidistribution of integer points on the sphere

For $D \in \mathbb{D}$ let

$$\mu_{\mathcal{I}_D} = \frac{1}{|\mathcal{I}_D|} \sum_{u \in \mathcal{I}_D} \delta_u.$$ 

Then $\mu_{\mathcal{I}_D}$ weak-* converges to $\mu_{S^2}$ as $D \to \infty$ in $\mathbb{D}$. 
The coronavirus

Figure: Covid-19
Numerical example

Figure: Integer points of norm 104851 projected onto $S^2$

Ellenberg, Michel, Venkatesh, *Linnik’s ergodic method and the distribution of integer points on spheres*
Linnik (1950-60’s)

Let $p > 2$ be prime and write

$$\mathbb{D}(p) = \{D \in \mathbb{D} : -D \in (\mathbb{F}_p^\times)^2\}.$$ 

Then $\mu_{\mathcal{L}_D} \xrightarrow{w^*} \mu_{S^2}$ as $D \to \infty$ in $\mathbb{D}(p)$.

Linnik’s condition $D \in \mathbb{D}(p)$ is equivalent to

*the stabilizer of a point in $\mathcal{R}_D$ is a split torus over $\mathbb{Q}_p$.*
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Basic idea of ergodic method: let $\nu$ be a weak-* limit.

1. Show that $\nu$ has maximal entropy, by bootstrapping an upper bound on the spacing of nearby points (Linnik’s Basic Lemma)

2. Apply uniqueness result of Einsiedler–Lindenstrauss.
Let $p > 2$ be prime and write

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Basic idea of ergodic method: let $\nu$ be a weak-* limit.

1. Show that $\nu$ has maximal entropy, by bootstrapping an upper bound on the spacing of nearby points (Linnik’s Basic Lemma)
2. Apply uniqueness result of Einsiedler–Lindenstrauss.

Quantitative version: $\mu_{\mathcal{A}_D} \xrightarrow{w^*} \mu_{\mathcal{S}_2}$ as $D \to \infty$ in the set

$$\left\{ D \in \mathbb{D} : \exists \ p \ll D^{\frac{1}{o(\log \log D)}} \text{ with } p \text{ split in } \mathbb{Q}(\sqrt{-D}) \right\}.$$ 

This set is all of $\mathbb{D}$ under GRH!
Conjecture A is true with a power savings rate: there is $\delta > 0$ such that for every “nice” $\Omega \subset S^2$ we have

$$\mu_{\mathcal{D}}(\Omega) = \mu_{S^2}(\Omega) + O(D^{-\delta}).$$

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\]

Spectral (automorphic) method: bound Weyl sums

\[
W(f, D) = \frac{1}{|\mathcal{S}_D|} \sum_{u \in \mathcal{S}_D} f(u),
\]

where \( f \in C(S^2) \) and \( \langle f, 1 \rangle = 0 \).

Enough to test on an orthonormal basis of \( L^2_0(S^2) \).

We take an orthonormal basis of arithmetic eigenfunctions. Recall

\[
S^2 = \text{SO}(3)/\text{SO}(2), \quad \text{SO}(3) = H^\times/\mathbb{R}^\times = G(\mathbb{R}),
\]

where \( G = PB^\times \) and \( B = B^{(2,\infty)} \).
Let $\Gamma = G(\mathbb{Z})$. Then $\Gamma$ acts on $\mathcal{R}_D$ by conjugation. Thus

$$W(f, D) = \frac{1}{|\Gamma \backslash \mathcal{I}_D|} \sum_{u \in \Gamma \backslash \mathcal{I}_D} F(u),$$

where

$$F(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} F(\gamma x) \quad \text{on} \quad S^2 = \Gamma \backslash S^2.$$

We have

$$S^2 = G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}})SO(2).$$
Let $\Gamma = G(\mathbb{Z})$. Then $\Gamma$ acts on $R_D$ by conjugation. Thus

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We have

$$S^2 = G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}})SO(2).$$

Take o.n.b. $\{f_i\}$ of $L^2_0(S^2)$ consisting of spherical harmonics

$$\Delta_{S^2} f_i = k(k + 1) f_i, \quad k \geq 1,$$

such that, upon adelization, the $\varphi_i$ on $S^2$ are joint eigenfunctions of the Hecke algebra $\mathcal{H}(G(\hat{\mathbb{Z}}) \backslash G(\mathbb{A}_f) / G(\hat{\mathbb{Z}}))$. 
Moreover, $\Gamma \backslash \mathcal{R}_D$ is a torsor for the class group $C_D$ of (an order in) $\mathbb{Q}(\sqrt{-D})$. Fixing a base point $u \in \mathcal{S}_D$ we have

$$W(f, D) = \frac{1}{h(-D)} \sum_{t \in C_D} F(t.u),$$

where $h(-D) = |C_D|$. This is an adelic toric integral: let

$$T_D = (\text{Res}_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}} \mathbb{G}_m)/\mathbb{G}_m.$$

Choosing $u \in \mathcal{S}_D$ yields an embedding $T_D \hookrightarrow G$. Let

$$T_D(\hat{\mathbb{Z}}) = T_D(\mathbb{A}_f) \cap G(\hat{\mathbb{Z}}) \quad \text{and} \quad T(\mathbb{R}) = g_\infty^{-1} \text{SO}(2)g_\infty.$$

Get an adelic toric orbit (finite collection of points)

$$Z_D = T_D(\mathbb{Q}) \backslash T_D(\mathbb{A})g_\infty / T_D(\hat{\mathbb{Z}}) T(\mathbb{R}) \hookrightarrow S^2.$$

Then $W(f; D) = \frac{1}{h(-D)} \int_{Z_D} \varphi$, where $\varphi$ is the adelization of $F$. 
Waldspurger (1985) et al.

Let $\sigma = \langle \varphi \rangle$ on $G = PB^\times$. Let $\pi = JL(\sigma)$ on $\text{PGL}_2$. Then

$$|W(f; D)|^2 \asymp D^{-1/2} \frac{L(1/2, \pi)L(1/2, \pi \times \eta_D)}{L(1, \eta_D)L(1, \text{Ad} \pi)}.$$  

Remark: If $f$ is of degree $k \geq 1$ then $\sigma_\infty = \text{sym}^{2k}$ on $G(\mathbb{R}) = \text{SO}(3)$ and $\pi_\infty = JL(\sigma_\infty) = D_{2k+2}$ on $\text{PGL}_2(\mathbb{R})$.

Siegel bound: We have $L(1, \eta_D) \gg_\epsilon D^{-\epsilon}$.

The problem is reduced to subconvex bounds on twists of $L$-functions by (quadratic) Dirichlet character twists.


There is $\delta > 0$ such that $L(1/2, \pi \times \eta_D) \ll D^{1/2-\delta}$.  

Second Linnik problem

For $D \in \mathbb{N}$ with $D \equiv 0, 3 \pmod{4}$ let

$$Q_D = \{AX^2 + BXY + CY^2 : \text{primitive, } B^2 - 4AC = -D\} / \text{SL}_2(\mathbb{Z}).$$

Let $Y(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ be the modular surface.

**Definition**

Put $\mathcal{H}_D = \{\text{unique root of } Q(X, 1) \text{ in } \mathbb{H} : Q \in Q_D\} \subset Y(1)$.

Let $\mu_{Y(1)}$ be the normalized hyperbolic measure on $Y(1)$.

**Conjecture B: Equidistribution of Heegner points on $Y(1)$**

For $D \in \mathbb{N}$ with $D \equiv 0, 3 \pmod{4}$ let

$$\mu_{\mathcal{H}_D} = \frac{1}{|\mathcal{H}_D|} \sum_{z \in \mathcal{H}_D} \delta_z.$$

Then $\mu_{\mathcal{H}_D}$ weak-* converges to $\mu_{Y(1)}$ along $D \equiv 0, 3 \pmod{4}$. 
### Linnik (1950’s-60’s)

Fix $p > 2$ a prime. Then $\mu_D \xrightarrow{w^*} \mu_Y(1)$ along $D \equiv 0, 3 \pmod{4}$ such that $-D \in (\mathbb{F}_p^\times)^2$.

Again, a quantitative version leads to Conjecture B under GRH.

### Duke (1988)

Conjecture B holds unconditionally, with a power savings rate.

Same ideas: Weyl sums $\rightarrow$ Waldspurger $\rightarrow$ Subconvexity

For $f \in C_0^\infty(Y(1))$ we wish to write the normalized Weyl sum

$$W(f; D) = \frac{1}{|\mathcal{H}_D|} \sum_{z \in \mathcal{H}_D} f(z) = \frac{1}{|h(-D)|} \sum_{t \in C_D} f(t.z_0)$$

where $z_0 \in \mathcal{H}_D$, as an adelic torus integral. We have

$$Y(1) = \text{PGL}_2(\mathbb{Q})\backslash \text{PGL}_2(\mathbb{A})/\text{PGL}_2(\hat{\mathbb{Z}})\text{SO}(2).$$
From $z_0 \in \mathcal{H}_D$ get embedding

$$T_D = (\text{Res}_{\mathbb{Q}(\sqrt{-D})/\mathbb{Q}} \mathbb{G}_m)/\mathbb{G}_m \hookrightarrow \text{PGL}_2.$$ 

Let $T_D(\hat{\mathbb{Z}}) = T_D(\mathbb{A}_f) \cap \text{PGL}_2(\hat{\mathbb{Z}})$ and $T(\mathbb{R}) = g^{-1}_\infty \text{SO}(2) g_\infty$. Get

$$Z_D = T_D(\mathbb{Q}) \backslash T_D(\mathbb{A}) g_\infty / T_D(\hat{\mathbb{Z}}) T(\mathbb{R}) \hookrightarrow Y(1).$$

Then $W(f; D) = \frac{1}{h(-D)} \int_{Z_D} \varphi$, where $\varphi$ is the adelization of $f$.

**Waldspurger (1985) et al.**

Let $\pi = \langle \varphi \rangle$ be cuspidal Maass on $\text{PGL}_2$. Then

$$|W(f; D)|^2 \asymp D^{-1/2} \frac{L(1/2, \pi) L(1/2, \pi \times \eta_D)}{L(1, \eta_D)^2 L(1, \text{Ad } \pi)}.$$ 

**Remark:** Here $\pi_\infty$ is a principal series representation on $\text{PGL}_2(\mathbb{R})$. The same subconvexity bound of DFI (1993) solves the problem.
Other variants

1) **Sparse equidistribution**: twisted Weyl sums, the numerator becomes $L(1/2, \pi \times \pi_\chi)$, subconvex bounds by Michel (2004)

2) Let $\mathbb{Q}(\sqrt{D})$ be real quadratic. Then $T_D \hookrightarrow \mathbb{P}B^\times$ for any indefinite $B$ such that $p$ split in $\mathbb{Q}(\sqrt{D})$ implies $B(\mathbb{Q}_p)$ split.

Obtain packets of closed geodesics on the unit tangent bundle of Shimura or modular curves.

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**Skubenko (1950-60’s)**

Equidistribution under Linnik’s condition.

**Duke (1987)**

Equidistribution for all positive fundamental discriminants.

Both proofs follow the same pattern.
Figure: $h(\mathbb{Q}(\sqrt{377})) = 1$

Einsiedler–Lindenstrauss–Michel–Venkatesh, *The distribution of closed geodesics on the modular surface, and Duke’s theorem*. Ergodic proof without congruence conditions! (torus split at $\infty$)
2) Let $G = PB^\times$, where $B = B^{(p,\infty)}$, where $p > 2$. Then

$$G(\mathbb{Q}) \backslash G(\mathbb{A})/G(\hat{\mathbb{Z}})G(\mathbb{R}) \simeq \text{Ell}^{ss}_p,$$

which has size $\frac{p-1}{12} + O(1)$ and has a natural probability measure

$$\mu_{\text{Ell}^{ss}_p}(e) = \frac{|\text{Aut}(e)|^{-1}}{\sum_{e' \in \text{Ell}^{ss}_p} |\text{Aut}(e')|^{-1}} = \frac{12}{p-1} |\text{Aut}(e)|^{-1}.$$
2) Let \( G = PB^\times \), where \( B = B^{(p,\infty)} \), where \( p > 2 \). Then

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G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}}) G(\mathbb{R}) \cong \text{Ell}^{ss}_p,
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\]

Let \( F \) be an imaginary quad field for which \( p \) is inert. Let \( H_F \) be the Hilbert class field of \( F \). For \( p | p \) get reduction map

\[
\text{Ell}^{\text{cm}}_{\mathcal{O}_F} \to \text{Ell}^{ss}_p, \quad E \mapsto E \mod p.
\]


**Michel (2004)**

Then the fibers of the reduction map are distributed according to \( \mu_{\text{Ell}^{ss}_p} \) as \( F \) varies over IQF for which \( p \) is inert, with power savings in the discriminant.
Simultaneous equidistribution

We return to the Linnik problems A and B.

Let $G_1 = PB^\times$, where $B = B^{(2,\infty)}$, and $G_2 = \text{PGL}_2$. For $D \in \mathbb{D}$:

$$G_1 \leftrightarrow T_D \leftrightarrow G_2,$$

simultaneous embeddings. We can then construct

$$\Delta : T_D \leftrightarrow G_1 \times G_2, \quad \Delta : Z_D \leftrightarrow S^2 \times Y(1).$$

Understand the distribution of $\Delta Z_D$ inside $S^2 \times Y(1)$ as $D \to \infty$.

**Expectation**

$\Delta Z_D$ should equidistribute to $S^2 \times Y(1)$ since the spaces $S^2$ and $Y(1)$ come from non-isomorphic quaternion algebras.
Classical description of $\Delta Z_D$

We have

$$Y(1) = \frac{\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})}{\text{SO}(2)} = \mathcal{L}_2,$$

where $\mathcal{L}_2$ is the space of unimodular lattices in $\mathbb{R}^2$ up to rotation.

Let $D \in \mathbb{D}$. For $v \in \mathcal{H}_D$ consider $\Lambda_v = \mathbb{Z}^3 \cap v^\perp$. Then

- rotate to a reference plane in $\mathbb{R}^3$,
- normalize to have covolume 1.

We obtain $[\Lambda_v] \in \mathcal{L}_2$. 
Classical description of $\Delta Z_D$

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- rotate to a reference plane in $\mathbb{R}^3$,
- normalize to have covolume 1.

We obtain $[\Lambda_v] \in \mathcal{L}_2$. Then

$$Z_D = \left\{ \left( \frac{v}{\|v\|}, [\Lambda_v] \right) : v \in \mathcal{R}(D) \right\} \subset S^2 \times Y(1).$$

So the question becomes:

**Does a primitive integral point on the sphere and the shape of its orthogonal lattice equidistribute in $S^2 \times Y(1)$?**

\( \Delta Z_D \) equidistributes to \( \mu S^2 \times \mu Y(1) \) as \( D \to \infty \) in \( \mathbb{D} \).

$\Delta Z_D$ equidistributes to $\mu_{S^2} \times \mu_{Y(1)}$ as $D \to \infty$ in $\mathbb{D}$.

Aka–Einsiedler–Shapira (2016)

Let $p, q > 2$ be distinct. Then $\Delta Z_D$ equidistributes to $\mu_{S^2} \times \mu_{Y(1)}$ as $D \to \infty$ in $\mathbb{D}(p, q) \cap \mathbb{F}$, where

$$\mathbb{D}(p, q) = \{ D \in \mathbb{D} : -D \in (\mathbb{F}_p^\times)^2, (\mathbb{F}_q^\times)^2 \}$$

and $\mathbb{F}$ is the set of square-free integers.

No quantification is available:

– no rate of equidistribution;

– their proof does not presently allow one to replace the congruence conditions by GRH.
Idea of proof of AES

Let $\nu$ be a weak-* limit.

1. Show that the push forward along both projections equidistributes in its copy.

2. Show, under the Linnik condition $D(p, q)$, that $\nu$ is invariant under $\text{Stab}_{\text{SO}_3(\mathbb{Q}_S)}(v_S)$, where $v_S \in \mathbb{Z}_S^3$ and $S = \{p, q\}$.

From (1) and (2) it follows that $\nu$ is a “joining”.


   a joining of higher rank torus actions is algebraic.

Since $G_1$ and $G_2$ are distinct, there is no non-trivial algebraic subgroup containing both $G_1$ and $G_2$. 
The proof is general and applies to all “hybrid situations”:

Aka–Luethi–Michel–Wieser (2020)

Let \( p_1, p_2, q_1, q_2 \) be distinct odd primes. The fibers of

\[
\text{Ell}^\text{cm}_{\mathcal{O}_F} \rightarrow \text{Ell}^\text{ss}_{p_1} \times \text{Ell}^\text{ss}_{p_2}, \quad E \mapsto (E \mod p_1, E \mod p_2)
\]

distribute according to \( \mu_{\text{Ell}^\text{ss}_{p_1}} \times \mu_{\text{Ell}^\text{ss}_{p_2}} \) as \( D \rightarrow +\infty \) in \( \mathbb{D}(q_1, q_1) \cap \mathbb{F} \)
such that \( p_1, p_2 \) are inert in \( \mathbb{Q}(\sqrt{-D}) \).
The proof is general and applies to all “hybrid situations”:

Let $p_1, p_2, q_1, q_2$ be distinct odd primes. The fibers of

$$\text{Ell}^{\text{cm}}_{\mathcal{O}_F} \to \text{Ell}^{\text{ss}}_{p_1} \times \text{Ell}^{\text{ss}}_{p_2}, \quad E \mapsto (E \mod p_1, E \mod p_2)$$

distribute according to $\mu_{\text{Ell}^{\text{ss}}_{p_1}} \times \mu_{\text{Ell}^{\text{ss}}_{p_2}}$ as $D \to +\infty$ in $\mathbb{D}(q_1, q_1) \cap \mathbb{F}$ such that $p_1, p_2$ are inert in $\mathbb{Q}(\sqrt{-D})$.

One can replace 2 copies by $n$ (pairwise non-isomorphic) copies, with a congruence condition for each copy.

For $Y(1) \times Y(1)$ there is also a mixing conjecture of Michel–Venkatesh, solved by Khayutin (2019) for $D \in \mathbb{D}(p, q) \cap \mathbb{F}$ and a Landau–Siegel zero assumption.
Main result: abstract set-up

Let $B_1, B_2/\mathbb{Q}$ be non-isomorphic, non-split, quaternion algebras.

Let $G_i = PB_i^\times$ and $G = G_1 \times G_2$.

Let $O_i$ be an Eichler order in $B_i(\mathbb{Q})$.

Let $K_f = K_1 \times K_2 \subset G(\mathbb{A}_f)$, where $K_i = PO_i^\times$.

Write $K = K_fK_\infty$ where $K_\infty = SO(2) \times SO(2) \subset G(\mathbb{R})$.

Put $X = G(\mathbb{Q}) \backslash G(\mathbb{A})/K$.

Let $F_d$ be a quadratic field extension of $\mathbb{Q}$ of discriminant $d$, optimally embedded in $O_i$.

Let $\Delta : T_d = (\text{Res}_{F_d/\mathbb{Q}} G_m)/G_m \hookrightarrow G$, the diagonal inclusion.

Let $g_\infty \in G(\mathbb{R})$ satisfy $g_\infty K_\infty g_\infty^{-1} = \Delta T_d(\mathbb{R})$.

Put $\Delta Z_D = G(\mathbb{Q}) \Delta T_d(\mathbb{A}) gK$, where $g = (1, g_\infty)$.
Main result: statement

Blomer – B. (in preparation)

Assume GRH. Then $\Delta Z_d$ equidistributes in $X$ with a logarithmic rate as $|d| \to \infty$: for every “nice” $\Omega \in X$ we have

$$\mu_{\Delta Z_d}(\Omega) = \mu_X(\Omega) + O_\epsilon((\log |d|)^{-1/4+\epsilon}).$$

Our proof goes through the theory of automorphic forms and Waldspurger’s theorem.

Plan for the remaining time:

1. describe a previous approach to this problem by R. Zhang;
2. motivate our different approach;
3. sketch our proof.
In the AES variant, the (unnormalized) Weyl sum is

$$S(\omega, \phi; D) = \sum_{\mathbf{v} \in \mathbb{Z}^3_{\text{prim}}, \|\mathbf{v}\| = D} \omega \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \phi(z_{\mathbf{v}}),$$

where $\omega$ is a spherical harmonic of degree $k$ on $S^2$ and $\phi$ is a Maass cusp form or unitary Eisenstein series on $Y(1)$.
In the AES variant, the (unnormalized) Weyl sum is

\[ S(\omega, \phi; D) = \sum_{v \in \mathbb{Z}_\text{prim}^3, \|v\| = D} \omega \left( \frac{v}{\|v\|} \right) \phi(z_v), \]

where \( \omega \) is a spherical harmonic of degree \( k \) on \( S^2 \) and \( \phi \) is a Maass cusp form or unitary Eisenstein series on \( Y(1) \).

R. Zhang (2015)

Let

\[ E(s, g, \omega, \phi) = \sum_{[\gamma] \in \Gamma_\infty \backslash \text{SL}_3(\mathbb{Z})} \omega(k(\gamma g))\phi(m(\gamma g))a(\gamma g)^{-s} \]

be the maximal Eisenstein series for \( \text{SL}_3(\mathbb{Z}) \) induced from \( \phi \) and transforming under \( K = \text{SO}(3) \) by \( \omega \). Then

\[ E(s, e, \omega, \phi) = \sum_{n \geq 1} S(\omega, \phi; n)n^{-s}. \]
Remarks

1. It is not clear from this description how GRH would imply any non-trivial bound on $S(\omega, \phi; D)$.
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R. Zhang (2015)

We have

$$\sum_{n \leq X} S(\omega, \phi; n) \ll \varepsilon X^{\frac{15}{14} + \varepsilon}.$$

Want to prove $S(\omega, \phi; D) = o(h(-D))$. This does not imply any bound on $S(\omega, \phi; D)$: they could exhibit cancellation on average.
Note that $T_d \subset G_1$ and $T_d \subset G_2$ are **Strong Gelfand pairs**

$$\forall \chi_p \in \widehat{T_d(\mathbb{Q}_p)} : \dim \text{Hom}_{T_d(\mathbb{Q}_p)}(\sigma_p, \chi_p) \leq 1.$$ 

This *multiplicity one* result lies at the heart of Waldspurger’s formula, in which the toric period squared is a *single L-function*. 

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But we have the following **Gelfand formation**:

$$G_1 \times G_2$$

$$\downarrow$$

$$T_d \times T_d$$

$$\downarrow$$

$$\Delta T_d$$

From which we expect to find a *family of L-functions*. 
Take $F_d$ IQF, $C_d$ its class group, $h_d = |C_d|$. The Weyl sum is

$$W(f_1, f_2; d) = \frac{1}{h_d} \sum_{t \in C_d} \Phi_1(t) \overline{\Phi_2(t)} \quad \text{(} \Phi_i(t) = \varphi_i(t.u_i)\text{).}$$

**Main estimate**

Under GRH, we have $W(f_1, f_2; d) \ll \epsilon \left( \log |d| \right)^{-1/4 + \epsilon}$. 
Take $F_d$ IQF, $C_d$ its class group, $h_d = |C_d|$. The Weyl sum is

$$W(f_1, f_2; d) = \frac{1}{h_d} \sum_{t \in C_d} \Phi_1(t)\overline{\Phi_2(t)} \quad (\Phi_i(t) = \varphi_i(t.u_i)).$$

**Main estimate**

Under GRH, we have $W(f_1, f_2; d) \ll \epsilon (\log |d|)^{-1/4 + \epsilon}$.

View as inner product on class group $C_d$. Plancherel formula gives

$$W(f_1, f_2; d) = \sum_{\chi \in \hat{C}_d} W_1(f_1, \chi; d)\overline{W_2(f_2, \chi; d)}.$$

Heuristic (under GRH):

- roughly $\approx |d|^{1/2}$ terms in the sum,
- each term is roughly $W_1(f_1, \chi; d)\overline{W_2(f_2, \chi; d)} \approx |d|^{-1/2}$

Might hope for square-root cancellation: $W(f_1, f_2; d) \ll |d|^{-1/4}$.
Crazy first step: void all cancellation!

\[ |W(f_1, f_2; d)| \leq \sum_{\chi \in \hat{C}_d} |W_1(f_1, \chi; d)W_2(f_2, \chi; d)|. \]
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\[ |W(f_1, f_2; d)| \leq \sum_{\chi \in \hat{C}_d} |W_1(f_1, \chi; d)W_2(f_2, \chi; d)|. \]

Assume \( W_1(f_1, \chi; d)W_2(f_2, \chi; d) \neq 0 \). Twisted Waldspurger gives

\[ |W_i(f_i, \chi; d)|^2 = |d|^{-1/2} \frac{L(1/2, \pi_i \times \chi)}{L(1, \eta_d)^2L(1, \text{Ad} \pi_i)}. \]

Get (using class number formula)

\[ |W(f_1, f_2; d)| \leq \mathcal{L}_d(1)S(d), \]

where \( \mathcal{L}_d(1) = L(1, \eta_d)^{-2}L(1, \text{Ad} \pi_1)^{-1/2}L(1, \text{Ad} \pi_2)^{-1/2} \) and

\[ S(d) = \frac{1}{h_d} \sum_{\chi \in \hat{C}_d} L(1/2, \pi_1 \times \chi)^{1/2}L(1/2, \pi_2 \times \chi)^{1/2}. \]
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Note that $\pi_1 \neq \pi_2$ since $G_1 \not\cong G_2$. Show $S(d) \ll \epsilon (\log |d|)^{-1/4+\epsilon}$. 
Pointwise GRH fails (as it must)

Cauchy-Schwartz reduces this to bounding

$$\frac{1}{h_d} \sum_{\chi \in \hat{C}_d} L(1/2, \pi \times \chi) \leq \max_{\chi \in \hat{C}_d} L(1/2, \pi \times \chi).$$

Clearly subconvexity is not going to do the job!

Under GRH (and Ramanujan), we have the general bound:

$$L(1/2, \pi) \ll \exp(A \log C(\pi)/ \log \log C(\pi))$$

Moreover (Soundararajan), there exist $d \in [X, 2X]$ such that

$$L(1/2, \eta_d) \gg \exp(c \sqrt{\log X} / \log \log X).$$

One can expect similar lower bounds on $L(1/2, \pi \times \chi)$ for $\chi \in \hat{C}_d$. 
Structurally similar situation: unipotent coefficients

QUE for arithmetic eigenfunctions (AQUE) on the modular surface.

– even weight (holomorphic): Holowinsky (2009):

\[ \frac{1}{T} \sum_{n \sim T} |\lambda_f(n)\lambda_f(n + 1)| \ll (\log T)^{-\delta} \]

– 1/2-integral weight (Maass): Lester–Radziwiłł (2019) on GRH:

\[ \frac{1}{T} \sum_{d \sim T} L(1/2, f \times \eta_d)^{1/2} L(1/2, f \times \eta_{d+1})^{1/2} \ll (\log T)^{-\delta}. \]

On average, these unipotent coefficients are of size \( \approx (\log n)^{-\delta} \), independently on small shifts.
Proof of main estimate

Let \( h = h_d \) and

\[
L_1(\chi) = L(1/2, \pi_1 \times \chi)^{1/2} \quad \text{and} \quad L_2(\chi) = L(1/2, \pi_2 \times \chi)^{1/2}.
\]

View \( \log L_1(\chi) \) as independent Gaussian random variables in \( \chi \).

Put \( L(\chi) = L_2(\chi)L_2(\chi) \).

Let \( \mu \) and \( \sigma^2 \) be the expectation and variance of \( \log L(\chi) \):

\[
\mu = \frac{\mu_1 + \mu_2}{2} \quad \text{and} \quad \sigma^2_{\text{naive}} = \frac{\sigma_1^2 + \sigma_2^2}{4}.
\]

Can calculate each \( \mu_i \) and \( \sigma^2_i \) under GRH: for small \( x \)

\[
\log L(1/2, \pi_i \times \chi) \lesssim \sum_{p \leq x} \frac{\lambda_{\pi_i}(p)a_\chi(p)}{p^{1/2}} + \frac{1}{2} \sum_{p^2 \leq x} \frac{\lambda_{\pi_i}(p^2)a_\chi(p^2)}{p} + \mu_i.
\]

The important feature is that \( \exp \left( \mu + \frac{\sigma^2_{\text{naive}}}{2} \right) \asymp (\log |d|)^{-1/4} \).
Proof (continued)

By partial summation we obtain

\[ S(d) = \frac{1}{h} \sum L(\chi) = \frac{1}{h} \int e^{V} \#\{\chi : \log L(\chi) > V\} dV \]

\[ = e^{\mu} \int e^{V} N(V) dV, \]

where

\[ N(V) = \frac{1}{h} \#\{\chi : \log L(\chi) - \mu > V\}. \]
Proof (continued)

By partial summation we obtain

\[ S(d) = \frac{1}{h} \sum_{\chi} L(\chi) = \frac{1}{h} \int_{\mathbb{R}} e^V \# \{ \chi : \log L(\chi) > V \} dV \]

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where

\[ N(V) = \frac{1}{h} \# \{ \chi : \log L(\chi) - \mu > V \}. \]

Now, for any \( k \geq 0 \), we have

\[ N(V) \leq V^{-2k} M_{2k}(V), \]

where

\[ M_{2k}(V) = \frac{1}{h} \sum_{\chi} \left( \log L(\chi) - \mu \right)^{2k}. \]
Proof (end)

By orthogonality of characters, we show, say for $k \gg \log \log |d|$, $M_{2k}(V) \ll \frac{(2k)!}{k!} \left( \frac{\sigma^2}{2} \right)^k$, where

$$
\sigma^2 = \sigma_{\text{naive}}^2 + \log L(1, \pi_1 \times \pi_2 \times \theta_d)^{1/2}.
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Since $\pi_1 \neq \pi_2$ this is well-defined!
Proof (end)

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$$N(V) \ll \frac{1}{V^{2k}} \frac{(2k)!}{k!} \left( \frac{\sigma^2}{2} \right)^k \asymp \left( \frac{2k \sigma^2}{eV^2} \right)^k \ll e^{-\frac{V^2}{2\sigma^2}}$$

upon choosing $k = V^2/(2\sigma^2)$. Get 

$$S(D) \ll e^\mu \int_{\mathbb{R}} e^{V - \frac{V^2}{2\sigma^2}} dV \asymp e^{\mu + \frac{1}{2}\sigma^2} \asymp (\log |d|)^{-1/4}. \quad \square$$

This approach dates back to Soundarajan (2009), on moments of the Riemann zeta function.
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Thank You!