SQG equation on the sphere
Analysis seminar, IAS

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Active scalars are a wide class of transport equations where the velocity is determined from the transported quantity in a certain way.
Active scalars are a wide class of transport equations where the velocity is determined from the transported quantity in a certain way. In this case a temperature $\theta$ evolves following

$$\begin{cases} 
\partial_t \theta(x, t) + u(x, t) \cdot \nabla \theta(x, t) = 0, \\
\nabla \cdot u(x, t) = 0 \\
u(x, t) = (-R_2 \theta, R_1 \theta)
\end{cases}$$

where $R_j$ denotes a Riesz potential

$$R_j(\theta)(x) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{y_j}{|y|^3} \theta(x - y) dy$$
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2. It is related to the 3D Euler equation.
A glimpse of history: the origins

**Constantin, Majda and Tabak.** Settled a connection with 3D Euler. Local existence and observe a possible scenario for finite time blow up. *Nonlinearity*, 1994.


In this case we consider

\[
\begin{cases}
\theta_t + u \cdot \nabla \theta + \kappa \Lambda^\alpha \theta = 0 \\
u = \nabla^\perp \Lambda^{-1} \theta \\
\theta(x, 0) = \theta_0(x), \kappa > 0
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The subcritical case, $\alpha > 1$, is well understood (Constantin-Wu, 1998). Global regularity for the critical case $\alpha = 1$ has been quite more challenging due to the possible balance between opposite strengths of the non-linear and the dissipative terms.
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1. Existence (local or global).
2. Uniqueness.
3. Finite time singularities (even in the compressible case).


Constantin, Vicol ($\mathbb{R}^2$): Nonlinear maximum principles for dissipative linear nonlocal operators and applications, GAFA. 2012.


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In fact all their proofs work in the $n$-dimensional tori or euclidean spaces. Our result deals only with the two dimensional sphere, namely:

Let \((M, g)\) be a compact orientable surface and \(g\) be a Riemannian metric the SQG in this case takes the form

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\begin{cases}
\frac{\partial \theta}{\partial t} + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\
u = \mathcal{R}_g \perp \theta = \nabla_g \perp \Lambda_g^{-1} \theta
\end{cases}
\]

where \(\Lambda_g = (-\Delta_g)^{\frac{1}{2}}\) and \(-\Delta_g\) is the Laplace-Beltrami operator.
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**Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)**

Let \(\theta_0 \in C^\infty(S^2)\) be the initial datum, then the solution remains smooth for any time \(t > 0\).
Given an initial datum $\theta_0 \in L^2(S^2)$ any weak solution becomes instantaneously continuous for any time $t > 0$. 

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Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

There is global well-posedness in \( H^s(\mathbb{S}^2) \) for any \( s > 3 \). In fact, any solution with such initial datum becomes smooth instantaneously.
The integral representation of the fractional laplacian as a singular integral is well known in the euclidean space $\mathbb{R}^n$ or the tori $\mathbb{T}^n$. 

\[ \Lambda_\alpha f(x) = c_{n,\alpha} \text{P.V.} \int \mathbb{R}^n f(x) - f(y) |x - y|^{n-\alpha} dy, \]

\[ \Lambda_\alpha f(x) = c_{n,\alpha} \text{P.V.} \sum_{\nu \in \mathbb{Z}^n} \int \mathbb{T}^n f(x) - f(y) |x - y - \nu|^n |x - y|^\alpha dy \]

for any $0 < \alpha < 2$. As an easy consequence of these one can get

Theorem (Córdoba-Córdoba inequality)

For any $\alpha \in (0, 2)$ and $f$ smooth enough, the following pointwise inequality holds

\[ f(x) \Lambda_\alpha f(x) \geq \frac{1}{2} \Lambda_\alpha (f^2)(x). \]
The integral representation of the fractional laplacian as a singular integral is well known in the euclidean space $\mathbb{R}^n$ or the tori $\mathbb{T}^n$. Indeed,

$$\Lambda^\alpha f(x) = c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy,$$

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Following the trajectories

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\frac{dx}{dt} = u(x, t)
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one gets

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\frac{d}{dt} (\theta(x(t), t)) = \theta_t + \nabla \theta \cdot \frac{dx}{dt} = 0
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deducing that \(\|\theta(\cdot, t)\|_{L^p}\) remains constant under the evolution \((1 \leq p \leq \infty)\).
Underlying particle dynamics

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deducing that \(|\theta(\cdot, t)|_{L^p} \) remains constant under the evolution (\(1 \leq p \leq \infty\)). It is easy to check, using the Córdoba-Córdoba inequality, that the \(L^p\) norms do not increase.
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**Theorem (A. Córdoba, A. D. M. 2015)**

*The following pointwise inequality holds*

\[
\frac{1}{2m} DN_{\Omega}(f^{2m})(x) \leq f(x)^{2m-1} DN_{\Omega}f(x)
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Proof: To begin with we propose the following Dirichlet problems in the domain

\[ \begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \\
u &= f \quad \text{in } \partial \Omega
\end{align*} \]

and

\[ \begin{align*}
\Delta v &= 0 \quad \text{in } \Omega \\
v &= f^{2m} \quad \text{in } \partial \Omega
\end{align*} \]

Then \( w = u^{2m} - v \) satisfies

\[ \begin{align*}
\Delta w &= 2m(2m - 1)|\nabla u|^2u^{2m-2} \quad \text{in } \Omega \\
w &= 0 \quad \text{in } \partial \Omega
\end{align*} \]
The Laplace-Beltrami operator, $\Delta_g$, in some local coordinates of the surface ($n = 2$) takes the form

$$\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j \right)$$

where $(g^{ij}) = (g_{ij})^{-1}$ is the inverse of the metric tensor.
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$$-\Delta_g Y_\ell = \lambda^2_\ell Y_\ell$$

where $\lambda_0 = 0$ and the eigenvalue increases to infinity as $\ell \in \mathbb{N}$ increases.
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where $\lambda_0 = 0$ and the eigenvalue increases to infinity as $\ell \in \mathbb{N}$ increases. The fractional Laplace-Beltrami operator acts on this basis as $(-\Delta_g)^{\alpha/2} Y_\ell = \lambda_\ell^\alpha Y_\ell$ and for any other function by linearity. Usual notation is $\Lambda_g^\alpha = (-\Delta_g)^{1/2}$. 
In the case of a general compact manifold (e.g. a sphere) the above was not available.

**Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)**

\[
\Lambda_g^\alpha f(x) = c_{n,\alpha} \text{P.V.} \int_M \frac{f(x) - f(y)}{d(x, y)^{n+\alpha}} (\chi u_0 + k_N) dy + E
\]

where \( k_N(x, y) = O(d(x, y)) \) is certain smooth function, \( \chi \) is a diagonal cutoff and the error gains derivatives

\[
E = O(\|f\|_{H^{-N}(M)}).
\]
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- Constantin-Vicol improvement with smoothing error.
- Fractional Sobolev embedding theorem for compact manifolds (cf. Aubin).
Recall from our previous sketch

**Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)**

Given an initial datum $\theta_0 \in L^2(S^2)$ any weak solution becomes instantaneously continuous for any time $t > 0$. 
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The instantaneous continuity result

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3. The argument does not work for dimensions greater than two.
De Giorgi’s technique 101

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**Lemma (Caccioppoli’s (energy) inequality)**

Let $u \geq 0$, $\Delta u \geq 0$ and $\varphi \in C_0^\infty(B_2)$ then

$$\int_{B_2} |\nabla(\varphi u)|^2(x)dx \leq C_\varphi \int_{B_2} u^2.$$
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\]

Using Sobolev’s embedding and this inequality one can prove

\[
E_k \leq C 2^{2k} E_{k-1}^{1+1/n}
\]

where

\[
E_k = \int_{B_1} (\varphi_k u_k)^2 \, dx
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where \( u_k = (u - (1 - 2^{-k}))_+ \) and \( \varphi_k \) is a cut off function on \( B_{1+2^{-k}} \).

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where \( u_k = (u - (1 - 2^{-k}))_+ \) and \( \varphi_k \) is a cut off function on \( B_{1+2^{-k}} \). Then \( E_k \to 0 \) if \( E_0 \) is small enough.
The previous idea can be used, this time

\[ E_k = \sup_{t \geq T_k} \int_M \theta_k^2 \, dx + 2 \int_{T_k} ^ \infty \int_M |\Lambda^{1/2} \theta_k|^2 \, dx \, dt \]
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\[ E_k = \sup_{t \geq T_k} \int_M \theta_k^2 dx + 2 \int_{T_k}^\infty \int_M |\Lambda^{1/2} \theta_k|^2 dx dt \]

to show

\[ E_k \leq C \frac{2^{k(1+2/n)+1}}{t_0 C^{2/n}} E_{k-1}^{1+1/n}. \]
Lemma (Local energy inequality, I)

Let \( \theta_k \) satisfy
\[
\partial_t \theta_k + u \cdot \nabla_g \theta_k \leq -\Lambda \theta_k
\]
and denote \( I(z_0) = [0, z_0] \). Let the function \( \eta \theta_k^*(x, t, z) \) be vanishing in \( M \times [0, \infty) \setminus B_g(h) \times I(z_0) \). Then if \( u \) satisfies
\[
\sup_{t \in (s, t)} \int_{B_g(h)} |u(x, t)|^{2n} dvol_g(x) \leq Ch^n
\]
and \( s \leq t \).
Lemma (Local energy inequality, II)

Then the following holds

\[
\begin{align*}
\int_s^t \int I(z_0) \int_{B_g(h)} |\nabla_{x,z}(\eta \theta_k^*)(x, t, z)|^2 \, dx \, dz \, dt &+ \int_{B_g(h)} (\eta \theta_k)^2(x, t) \, dx \\
&\leq C \left\{ \int_{B_g(h)} (\eta \theta_k)^2(x, s) \, dx + h \int_s^t \int_{B_g(h)} |\nabla_x \eta \theta_k|^2 \, dx \, dt \\
&\quad + \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z} \eta \theta_k^*|^2 \, dvol_g(x) \, dz \, dt \\
&\quad + \int_s^t \int_{B_g(h)} (\eta \theta_k)^2(x, t) \, dvol_g(x) \, dt \right\}.
\end{align*}
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Apart from this one needs

Fractional Sobolev embedding to get global $L^\infty(M)$ control out of $L^2$ norm.

Nice barrier functions satisfying certain properties (recall no scaling!).

Very tricky induction argument to get oscillation decay using the above (for small energy).

Isoperimetric inequality on the sphere (cf. De Giorgi’s to get rid of the small energy assumption).

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- Isoperimetric inequality on the sphere (cf. De Giorgi’s to get rid of the small energy assumption).
- Rotations to get the logarithmic modulus of continuity.
Lemma

Let $f$ be a smooth function on the sphere $\mathbb{S}^2$ and $0 < \alpha < 2$. Then provided that $|\nabla_g f(x)| \geq C\|f\|_\infty$ we have the pointwise bound

$$\nabla_g f(x) \cdot \nabla_g \Lambda^\alpha f(x) \geq \frac{1}{2} \Lambda^\alpha (|\nabla_g f|^2)(x) + \frac{1}{4} D(x) + \frac{|\nabla_g f(x)|^{2+\alpha}}{C\|f\|_\infty^\alpha} + O$$

where $D$ denotes some functional (defined in the proof), $O = O(\|\nabla_g f\|_\infty^2)$ is an error term and the constant $C$ depends on $\alpha$ but is independent of $x$. 

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Non linear maximum principle
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Careful: notice the non commutativity of some operators (despite $\Psi DO$). We need stereographic projection, integral representation with smoothing error.
Let \( L = (\partial_t + u \cdot \nabla g + \Lambda_g) \).
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\[
\frac{1}{2} L(|\nabla g \theta|^2)(x) + \frac{1}{c \| \theta \|_\infty} |\nabla g \theta(x)|^3 \leq C |\nabla g \theta(x)|^2 + O(\| \nabla g \theta \|_\infty^2)
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holds for any \( t > 0 \).
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holds for any \( t > 0 \). Evaluating formally at \( \bar{x} \), a point that reaches the maximum of \(|\nabla g \theta|(x)|\), one gets

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Heuristically this prevents indefinite growth for the $L^\infty$ norm of the gradient.
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$$\frac{d}{dt}|\nabla g \theta|^2(\bar{x}) \leq 0$$

Heuristically this prevents indefinite growth for the $L^\infty$ norm of the gradient. This heuristic can be replaced by a honest argument.
From the above it follows that

$$\| \nabla_g \theta \|_{L^\infty(M)} < +\infty$$

in fact, it will be bounded by some absolute constant $C > 0$. 
From the above it follows that

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in fact, it will be bounded by some absolute constant \( C > 0 \). This control is all one needs to prove the theorem thanks to the usual energy estimates.

**Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)**

*There is global well-posedness in \( H^s(\mathbb{S}^2) \) for any \( s > 3 \). In fact, any solution with such initial datum becomes smooth instantaneously.*
Thank you!