Meridional essential surfaces of unbounded Euler characteristics in knot complements

João M. Nogueira
University of Coimbra, Portugal

Institute for Advanced Study
April 20, 2016
Essential surfaces have a preeminent presence on the understanding of 3-manifold topology, and of knot complements in particular.

One interesting phenomenon is certain knot complements having essential surfaces of arbitrarily large Euler characteristics.
Essential surfaces have a preeminent presence on the understanding of 3-manifold topology, and of knot complements in particular.

One interesting phenomenon is certain knot complements having essential surfaces of arbitrarily large Euler characteristics.
Unbounded Euler characteristic

Lyon, 71:
- Knot complement with closed essential surfaces of arbitrarily high genus;
- Knot with incompressible Seifert surfaces of arbitrarily high genus.

Later, other examples with the same properties for closed surfaces were also obtained, for instance: Oertel, 84; Eudave-Muñoz and Neumann-Coto, 04; Y. Li, 09.
Unbounded Euler characteristic

Lyon, 71:

- Knot complement with closed essential surfaces of arbitrarily high genus;
- Knot with incompressible Seifert surfaces of arbitrarily high genus.

Later, other examples with the same properties for closed surfaces were also obtained, for instance: Oertel, 84; Eudave-Muñoz and Neumann-Coto, 04; Y. Li, 09.
Unbounded Euler characteristic

Question:

Can the arbitrarily large Euler characteristic property be obtained from the number of boundary components?

Or simultaneously from the genus and the number of boundary components?

We answer affirmatively to these questions today.
Unbounded Euler characteristic

Question:

Can the arbitrarily large Euler characteristic property be obtained from the number of boundary components?

Or simultaneously from the genus and the number of boundary components?

We answer affirmatively to these questions today.
Arbitrarily high number of boundaries

We start by considering meridional essential planar surfaces or, equivalently, essential tangle decompositions of knots.
$n$-string essential tangle

$n$-string tangle:
$(B, s_1 \cup \cdots \cup s_n)$, is a ball with $n$ p.e. disjoint arcs.

Essential:
$(n \geq 2)$ if there is no disk, in $B - \bigcup_is_i$, separating the collection of arcs;
$(n=1)$ if the single string is knotted.
$n$-string essential tangle

$n$-string tangle: $(B, s_1 \cup \cdots \cup s_n)$, is a ball with $n$ p.e. disjoint arcs.

**Essential:**
$(n \geq 2)$ if there is no disk, in $B - \bigcup_i s_i$, separating the collection of arcs;
$(n=1)$ if the single string is knotted.

Not essential
**n-string essential tangle**

**n-string tangle:**
\( (B, s_1 \cup \cdots \cup s_n) \), is a ball with \( n \) p.e. disjoint arcs.

**Essential:**
\((n \geq 2)\) if there is no disk, in \( B - \bigcup_i s_i \), separating the collection of arcs;
\((n=1)\) if the single string is knotted.
$n$-string tangle decompositions

$n$-string tangle decomposition:

$$(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$$

with $T_i$ being $n$ arcs.

Essential: if both $(B_i, T_i)$ are essential.
$n$-string tangle decompositions

$n$-string tangle decomposition:
$$(\mathbb{S}^3, K) = (B_1, T_1) \cup_S (B_2, T_2)$$

with $T_i$ being $n$ arcs.

**Essential**: if both $(B_i, T_i)$ are essential.
Knots and their essential tangle decompositions

Lickorish, 81: If a knot has a 2-string essential tangle decomposition with no local knots then the knot is prime.

Ozawa, 97: If a knot has a 2-string essential free tangle decomposition then this is the unique essential tangle decomposition of the knot up to isotopy.

Knots with no closed essential surfaces (CGLS, 87), tunnel number one knots (Gordon and Reid, 95), and free genus one knots (Matsuda and Ozawa, 98), have no essential tangle decompositions.
Knots and their essential tangle decompositions

Lickorish, 81: If a knot has a 2-string essential tangle decomposition with no local knots then the knot is prime.

Ozawa, 97: If a knot has a 2-string essential free tangle decomposition then this is the unique essential tangle decomposition of the knot up to isotopy.

Knots with no closed essential surfaces (CGLS, 87), tunnel number one knots (Gordon and Reid, 95), and free genus one knots (Matsuda and Ozawa, 98), have no essential tangle decompositions.
Knots and their essential tangle decompositions

Lickorish, 81: If a knot has a 2-string essential tangle decomposition with no local knots then the knot is prime.

Ozawa, 97: If a knot has a 2-string essential free tangle decomposition then this is the unique essential tangle decomposition of the knot up to isotopy.

Knots with no closed essential surfaces (CGLS, 87), tunnel number one knots (Gordon and Reid, 95), and free genus one knots (Matsuda and Ozawa, 98), have no essential tangle decompositions.
Knots and their essential tangle decompositions

Mizuma and Tsutsumi, 08: The number of strings that are not parallel to other strings in an essential tangle decomposition of a fixed knot is bounded.

Question:

Is the number of strings on essential tangle decompositions of a fixed knot bounded?
Knots and their essential tangle decompositions

Mizuma and Tsutsumi, 08: The number of strings that are not parallel to other strings in an essential tangle decomposition of a fixed knot is bounded.

Question:

Is the number of strings on essential tangle decompositions of a fixed knot bounded?
Essential tangle decompositions: arbitrarily high number of strings

Theorem 1 (N., 15).
There is an infinite collection of prime knots such that for all $n \geq 2$
each knot has a $n$-string essential tangle decomposition.

Corollary.
There is an infinite collection of knots such that for all $n \geq 1$
each knot has a $n$-string essential tangle decomposition.
Strategy for the proof

Handlebody of genus 2

\[ \rightarrow \]

embedding
into \( S^3 \)

Handlebody-knot \( 4_1 \)
Strategy for the proof

Connected sum of two trefoils

embedding into $S^3$

Handlebody-knot $4_1$
Strategy for the proof

Connected sum of two trefoils

embedding into $S^3$

Spine of 4

Strategy for the proof

Connected sum of two trefoils
Strategy for the proof

Connected sum of two trefoils

embedding into $S^3$
Strategy for the proof

Arbitrarily high number of boundaries
Arbitrarily high genus
Arbitrarily high genus and number of boundaries

(embedding into $S^3$)
Strategy for the proof

Arbitrarily high number of boundaries
Arbitrarily high genus
Arbitrarily high genus and number of boundaries
Strategy for the proof

Arbitrarily high number of boundaries
Arbitrarily high genus
Arbitrarily high genus and number of boundaries

embedding into $S^3$
Strategy for the proof

Arbitrarily high number of boundaries
Arbitrarily high genus
Arbitrarily high genus and number of boundaries
Strategy for the proof

Meridional essential surfaces

Introduction

Arbitrarily high number of boundaries

Arbitrarily high genus

Arbitrarily high genus and number of boundaries

Embedding into $S^3$
Strategy for the proof

Arbitrarily high number of boundaries
Arbitrarily high genus
Arbitrarily high genus and number of boundaries

Meridional essential surfaces
Strategy for the proof

Arbitrarily high number of boundaries
Arbitrarily high genus
Arbitrarily high genus and number of boundaries

embedding into $S^3$
Knots with essential tangle decompositions with arbitrarily high number of strings
Remarks and questions

Infinite collection of prime knots such that each knot has a meridional planar essential surface in its complement for arbitrarily high number of boundary components.

The examples are satellite knots. There are hyperbolic knot examples with the same property obtained from similar ideas.

What characterizes knots with essential tangle decompositions with (without) arbitrarily high number of strings?

Are there knots with meridional essential surfaces of arbitrarily high genus on top of arbitrarily high number of boundary components? (Following sections.)
Remarks and questions

Infinite collection of prime knots such that each knot has a meridional planar essential surface in its complement for arbitrarily high number of boundary components.

The examples are satellite knots. There are hyperbolic knot examples with the same property obtained from similar ideas.

What characterizes knots with essential tangle decompositions with (without) arbitrarily high number of strings?

Are there knots with meridional essential surfaces of arbitrarily high genus on top of arbitrarily high number of boundary components? (Following sections.)
Remarks and questions

Infinite collection of prime knots such that each knot has a meridional planar essential surface in its complement for arbitrarily high number of boundary components.

The examples are satellite knots. There are hyperbolic knot examples with the same property obtained from similar ideas.

What characterizes knots with essential tangle decompositions with (without) arbitrarily high number of strings?

Are there knots with meridional essential surfaces of arbitrarily high genus on top of arbitrarily high number of boundary components? (Following sections.)
Knot exterior with meridional essential surfaces of unbounded genus

Lyon, 71: Knot exterior with closed essential surfaces of arbitrarily high genus.

Question:

Is there a prime knot exterior with meridional essential surfaces of arbitrarily high genus and two boundary components?
Knot exterior with meridional essential surfaces of unbounded genus

Lyon, 71: Knot exterior with closed essential surfaces of arbitrarily high genus.

Question:

Is there a prime knot exterior with meridional essential surfaces of arbitrarily high genus and two boundary components?
Knot exterior with meridional essential surfaces of unbounded genus

**Theorem 2 (N., 15).** There is an infinite collection of prime knots such that for all \( g \geq 1 \) each knot has a meridional essential surface of genus \( g \) and two boundary components.

**Corollary.** There is an infinite collection of knots such that for all \( g \geq 0 \) each knot has a meridional essential surface of genus \( g \) and two boundary components.
Knot exterior with meridional essential surfaces of unbounded genus

We should avoid:

- satellite construction to obtain primeness, as we want the surfaces to have only two boundary components;
- use connected sum as the base for the construction, as we want the knots to be prime.

Strategy:
Construct the knot by identifying the boundaries of two solid tori containing, each, a properly embedded essential arc.
Knot exterior with meridional essential surfaces of unbounded genus

We should avoid:
- satellite construction to obtain primeness, as we want the surfaces to have only two boundary components;
- use connected sum as the base for the construction, as we want the knots to be prime.

Strategy:
Construct the knot by identifying the boundaries of two solid tori containing, each, a properly embedded essential arc.
Knot exterior with meridional essential surfaces of unbounded genus
Knot exterior with meridional essential surfaces of unbounded genus
Knot exterior with meridional essential surfaces of unbounded genus

Lemma

- The surfaces $\partial H$ and $\partial T$ are incompressible in $E_H(T)$;
- The arc $s$ is essential in $T$;
- There is no properly embedded essential disk in $E_H(T)$ with boundary the union of an arc in $\partial T$ and an arc $\partial H$, and not bounding a disk in $\partial E_H(T)$. 
Knot exterior with meridional essential surfaces of unbounded genus

Lemma

- The surfaces $\partial H$ and $\partial T$ are incompressible in $E_H(T)$;
- The arc $s$ is essential in $T$;
- There is no properly embedded essential disk in $E_H(T)$ with boundary the union of an arc in $\partial T$ and an arc $\partial H$, and not bounding a disk in $\partial E_H(T)$. 
Knot exterior with meridional essential surfaces of unbounded genus

The surfaces $F_g$:
Knot exterior with meridional essential surfaces of unbounded genus

The surfaces $F_g$:
Knot exterior with meridional essential surfaces of unbounded genus

The surfaces $F_g$:
Knot exterior with meridional essential surfaces of unbounded genus

The surfaces $F_g$: 
Knot exterior with meridional essential surfaces of unbounded genus

Branched surface:
Knot exterior with meridional essential surfaces of unbounded genus

Branched surface:
Knot exterior with meridional essential surfaces of unbounded genus

Branched surface:
Knot exterior with meridional essential surfaces of unbounded genus

Branched surface:
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

We now know:

1. There is a knot exterior with meridional planar surfaces for arbitrarily high number of boundary components.
2. There is a knot exterior with meridional essential surfaces for arbitrarily high genus and two boundary components.

Question:

Can the genus and number of boundary components of meridional essential surfaces be simultaneously unbounded?
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

We now know:

1. There is a knot exterior with meridional planar surfaces for arbitrarily high number of boundary components.
2. There is a knot exterior with meridional essential surfaces for arbitrarily high genus and two boundary components.

Question:

Can the genus and number of boundary components of meridional essential surfaces be simultaneously unbounded?
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Menasco, 84: In the complement of an alternating knot, for a fixed number of boundary components there are finitely many meridional essential surfaces.

**Proposition.** Let $K$ be a knot which complement contains a meridional essential surface of genus $g$ and $n$ boundary components.
Then, there is a knot $h(K)$ which complement contains meridional essential surfaces of genus $g$ and $b$ boundary components for all even $b \geq 2n$. 
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Menasco, 84: In the complement of an alternating knot, for a fixed number of boundary components there are finitely many meridional essential surfaces.

**Proposition.** Let $K$ be a knot which complement contains a meridional essential surface of genus $g$ and $n$ boundary components. Then, there is a knot $h(K)$ which complement contains meridional essential surfaces of genus $g$ and $b$ boundary components for all even $b \geq 2n$. 
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Proposition.** Let $K$ be a knot which complement contains a meridional essential surface of genus $g$ and $n$ boundary components. Then, there is a knot $h(K)$ which complement contains meridional essential surfaces of genus $g$ and $b$ boundary components for all even $b \geq 2n$. 
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Proposition.** Let $K$ be a knot which complement contains a meridional essential surface of genus $g$ and $n$ boundary components. Then, there is a knot $h(K)$ which complement contains meridional essential surfaces of genus $g$ and $b$ boundary components for all even $b \geq 2n$. 
Sketch of proof

Connected sum of two knots with meridional essential surfaces with two boundary components and unbounded genus
Sketch of proof

Connected sum of two knots with meridional essential surfaces with two boundary components and unbounded genus

Handlebody-knot $\Gamma$
**Sketch of proof**

Connected sum of two knots with meridional essential surfaces with two boundary components and unbounded genus

The knot $h(K)$ results from a satellite of $\Gamma$ with companion $K$. 

Handlebody-knot $\Gamma$
Sketch of proof

Connected sum of two knots with meridional essential surfaces with two boundary components and unbounded genus

The knot $h(K)$ results from a satellite of $\Gamma$ with companion $K$.

The branched surface carrying with positive weights the resulting meridional essential surfaces in the complement of $h(k)$.

Handlebody-knot $\Gamma$
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Theorem 3 (N., 15): There are infinitely many knots each of which having in its exterior meridional essential surfaces of all genus and $2n$ boundary components for all $n \geq 1$. 
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Theorem 3 (N., 15): There are infinitely many knots each of which having in its exterior meridional essential surfaces of all genus and $2n$ boundary components for all $n \geq 1$.

Choosing the knot $K$ to have meridional essential surfaces of all genus and two boundary components, the satellite $h(K)$ is as in the statement.
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Theorem 4 (N., 15):** There are infinitely many hyperbolic knots each of which having in its exterior meridional essential surfaces of simultaneously unbounded genus and number of boundary components.
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Sketch of proof for hyperbolic knots:** Consider a knot $K$ as in the statement of Theorem 3.

From Myers, 82, let $J$ be a null-homotopic knot in $E(K)$ with hyperbolic complement.

We proceed with $\frac{1}{r}$-Dehn filling on $J$ and get a hyperbolic knot $K_r \subset S^3$.

From Boileau-Wang, 96, there is a degree-one map $f : E(K_r) \to E(K)$. This map preserves the meridional boundaries of the surfaces.

The restriction to surfaces $f : F_{g;b} \to S_{g;b}$ is also a degree-one map. From Edmonds, 79, it is a pinch map. The genus of $F_{g;b}$ is higher than of $S_{g;b}$.
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Sketch of proof for hyperbolic knots:** Consider a knot $K$ as in the statement of Theorem 3.

From Myers, 82, let $J$ be a null-homotopic knot in $E(K)$ with hyperbolic complement.

We proceed with $\frac{1}{r}$-Dehn filling on $J$ and get a hyperbolic knot $K_r \subset S^3$.

From Boileau-Wang, 96, there is a degree-one map $f : E(K_r) \to E(K)$. This map preserves the meridional boundaries of the surfaces.

The restriction to surfaces $f : F_{g;b} \to S_{g;b}$ is also a degree-one map. From Edmonds, 79, it is a pinch map. The genus of $F_{g;b}$ is higher than of $S_{g;b}$. 
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Sketch of proof for hyperbolic knots:** Consider a knot $K$ as in the statement of Theorem 3.

From Myers, 82, let $J$ be a null-homotopic knot in $E(K)$ with hyperbolic complement.

We proceed with $\frac{1}{r}$-Dehn filling on $J$ and get a hyperbolic knot $K_r \subset S^3$.

From Boileau-Wang, 96, there is a degree-one map $f : E(K_r) \to E(K)$. This map preserves the meridional boundaries of the surfaces.

The restriction to surfaces $f : F_{g;b} \to S_{g;b}$ is also a degree-one map. From Edmonds, 79, it is a pinch map. The genus of $F_{g;b}$ is higher than of $S_{g;b}$.
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Sketch of proof for hyperbolic knots:** Consider a knot $K$ as in the statement of Theorem 3.

From Myers, 82, let $J$ be a null-homotopic knot in $E(K)$ with hyperbolic complement.

We proceed with $\frac{1}{r}$-Dehn filling on $J$ and get a hyperbolic knot $K_r \subset S^3$.

From Boileau-Wang, 96, there is a degree-one map $f : E(K_r) \to E(K)$. This map preserves the meridional boundaries of the surfaces.

The restriction to surfaces $f : F_{g;b} \to S_{g;b}$ is also a degree-one map. From Edmonds, 79, it is a pinch map. The genus of $F_{g;b}$ is higher than of $S_{g;b}$. 
Knot exterior with meridional essential surfaces of unbounded genus and boundary components

**Sketch of proof for hyperbolic knots:** Consider a knot $K$ as in the statement of Theorem 3.

From Myers, 82, let $J$ be a null-homotopic knot in $E(K)$ with hyperbolic complement.

We proceed with $\frac{1}{r}$-Dehn filling on $J$ and get a hyperbolic knot $K_r \subset S^3$.

From Boileau-Wang, 96, there is a degree-one map $f : E(K_r) \to E(K)$. This map preserves the meridional boundaries of the surfaces.

The restriction to surfaces $f : F_{g;b} \to S_{g;b}$ is also a degree-one map. From Edmonds, 79, it is a pinch map. The genus of $F_{g;b}$ is higher than of $S_{g;b}$. 
Meridional essential surfaces of unbounded Euler characteristics in knot complements

João M. Nogueira
University of Coimbra, Portugal

Institute for Advanced Study
April 20, 2016