Projective Dehn twist

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Symplectic manifolds

A symplectic manifold \((M^{2n}, \omega)\) is

- a smooth manifold \(M\)
- equipped with \(\omega \in \Omega^2(M)\) such that \(d\omega = 0\), \(\omega^n\) nowhere vanishing

eg. \((\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)\), \((T^*Q, \omega_{can})\), etc
A symplectomorphism $\phi : (M, \omega_M) \rightarrow (N, \omega_N)$ is

- a diffeomorphism $\phi : M \rightarrow N$, such that

$\phi^* \omega_N = \omega_M$

eg. when $(M, \omega_M) = (N, \omega_N)$ compact, time 1 flow along a vector field $X$ on $M$ such that $\mathcal{L}_X \omega_M = 0$
Dehn twist

Given a Lagrangian sphere $S \subset (M, \omega)$, one can perform Dehn twist $\tau_S : (M, \omega) \rightarrow (M, \omega)$ which is a symplectomorphism.

Figure: $S$ is purple, $L$ and $\tau_S(L)$ are green
Fukaya category

Fukaya category $\mathcal{F}uk(M, \omega)$ is an $A_\infty$ category

- objects: Lagrangians submanifolds $L$ (with additional structures/restrictions)
- morphism: $\text{hom}(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{K} \langle p \rangle$
- $A_\infty$ operations
  $\mu_k : \text{hom}(L_{k-1}, L_k) \otimes \cdots \otimes \text{hom}(L_0, L_1) \to \text{hom}(L_0, L_k)$
Quasi-equivalence

Given an $A_\infty$ category $\mathcal{A}$, one can take its cohomological category $H(\mathcal{A})$.
An $A_\infty$ functor $F : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if the induced functor on the cohomological category is an equivalence.
Theorem (Seidel)

For any graded compact exact Lagrangian $L$, there is an exact triangle in $D^\pi \mathcal{F}uk(M, \omega)$

$$HF(S, L) \otimes S \rightarrow L \rightarrow \tau_S(L) \rightarrow HF(S, L) \otimes S[1]$$

The induced autoequivalence $T_S : D^\pi \mathcal{F}uk(M, \omega) \rightarrow D^\pi \mathcal{F}uk(M, \omega)$ by $\tau_S$ can be formulated purely algebraically.
Let $X$ be a smooth projective variety. An object $\mathcal{E}$ in $D^b(X)$ is spherical if

- $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$, and
- $\text{Ext}^*(\mathcal{E}, \mathcal{E}) = H^*(S^{\dim(X)})$

A spherical object determines an autoequivalence $T_{\mathcal{E}}$ on $D^b(X)$. 
Table: From symplectomorphism to autoequivalence

<table>
<thead>
<tr>
<th>$(M, \omega)$</th>
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<th>$D^b(X)$</th>
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<tbody>
<tr>
<td>$S$</td>
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<td>$\mathcal{E}$</td>
</tr>
<tr>
<td>$\tau_S$</td>
<td>$T_S$</td>
<td>$T_\mathcal{E}$</td>
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Questions

- Are there other symplectomorphisms supported near Lagrangian submanifolds?
- What can one say about the induced auto-equivalences?
A parallel story for projective space:

**Table**: From symplectomorphism to autoequivalence

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</tr>
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<td>$\tau_P$</td>
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Here, $P$ is a Lagrangian (real/complex) projective space and $\mathcal{P}$ is a $\mathbb{P}$-object in $D^b(X)$. 
The definition of $\mathbb{P}$-object and $\mathbb{P}$-twist is due to Huybrechts-Thomas and is motivated by the symplectomorphism $\tau_P$. However, the relation between $\tau_P$ and $T_P$ is still conjectural.

**Conjecture (Huybrechts-Thomas)**

The induced autoequivalence on $D^\pi\mathcal{F}uk(M,\omega)$ by $\tau_P$ is $T_P$. 
Partial result

In monotone setting, using Mau-Wehrheim-Woodward functor and Biran-Cornea Lagrangian cobordism theory, we have

**Theorem (M-Wu)**

\[
\tau_P(L) = \text{Cone}(\text{Cone}(\text{hom}(P, L) \otimes P[−2] \to \text{hom}(P, L) \otimes P) \to L)
\]

for every L

It looks similar to \( T_P(L) \) but the morphisms in the theorem are not explicitly determined.
Question

Are spherical twists and $\mathbb{P}$-twists related?
A hybrid

A Lagrangian $S^2 = \mathbb{CP}^1$ in $D^\pi\mathcal{F}uk(M, \omega)$ is both spherical and projective (similarly $S^1 = \mathbb{RP}^1$)

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A Lagrangian $P = \mathbb{RP}^{2n+1}$ is

- a $\mathbb{P}$-object when $\text{char}(K) = 2$
- a spherical object when $\text{char}(K) \neq 2$

**Question**

What is the induced autoequivalence of $\tau_P$ on $D^\pi \mathcal{F}uk(M, \omega)$ when $\text{char}(K) \neq 2$?
When $\text{char}(\mathbb{K}) \neq 2$, the induced autoequivalence by $\tau_P$ on $D^\pi \mathcal{F}uk(M, \omega)$ is

- well-defined when $P$ is (relatively) spin
- not a projective twist
- not a spherical twist
- not a square of a spherical twist
Work in progress

Theorem (M-Wu)

Let \( P = \mathbb{RP}^{4n+3} \) be a monotone Lagrangian in a close monotone \((M, \omega)\). The induced autoequivalence by \( \tau_P \) is a simultaneous spherical twist by two spherical objects when \( \text{char}(\mathbb{K}) \neq 2 \).

Goal:
- Explain to you autoequivalences obtained by \( S^n/\Gamma \) twists in various characteristic
- Observe new phenomena of autoequivalences of \( D^b(X) \)
THANK YOU

Thank you very much for your attention!