Vertex algebras and quantum master equation

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Motivation: Quantum B-model

A-model (symplectic) \overset{\text{mirror}}{\leftrightarrow} B-model (complex)

Gromov-Witten type theory

- counting genus zero curves

Hodge type theory

- Variation of Hodge structures
- counting higher genus curves
- ?

Question

What is the geometry of higher genus B-model? In other words, what is the quantization of VHS on CY geometry?
Consider
\[ e^{u/t} = 1 + \frac{u}{t} + \sum_{k \geq 1} \frac{1}{(k + 1)!} \frac{u^{k+1}}{t^{k+1}}. \]
Motivation: Integrable hierarchy

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e^{u/t} = 1 + \frac{u}{t} + \sum_{k \geq 1} \frac{1}{(k + 1)!} \frac{u^{k+1}}{t^{k+1}}.
\]

Promote \( u \) to be a field \( u(x) \), introduce the Poisson bracket
\[
\{u(x), u(y)\} = \partial_x \delta(x - y).
\]

Then we find infinite number of pairwise commuting Hamiltonians
\[
h_k = \frac{1}{(k + 1)!} \int dx u^{k+1}, \quad k \geq 1.
\]

This is the dispersionless KdV integrable hierarchy.
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Question
Why integrable hierarchies in topological string? Is the exponential map \( e^{\frac{u}{t}} \) universal?
We will be focused on the *B-twisted* topological string.

- [Bershadsky-Cecotti-Ooguri-Vafa, 1994]: B-model on CY three-fold can be described by a gauge theory
  
  → **Kodaira-Spencer gauge theory.**

This describes the *leading cubic vertex* of Zwiebach’s string field action in the topological B-model.
We will be focused on the $B$-twisted topological string.

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- [Costello-L, 2012]: The full description of Zwiebach’s string field action in the B-model on arbitrary CY geometry

  → **BCOV theory.**
Motivation revisited

Higher genus B-model is described by the quantization of BCOV theory in the Batalin-Vilkovisky formalism.
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$$X \times \Sigma \to \Sigma, \quad \Sigma = \mathbb{C}, \mathbb{C}^*, \text{ or } E.$$ 

Start with B-model on $X \times \Sigma$, and compactify along $X$

$$\implies \text{effective 2d chiral QFT on } \Sigma.$$
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1. Higher genus B-model is described by the quantization of BCOV theory in the Batalin-Vilkovisky formalism.

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$\implies$ effective 2d chiral QFT on $\Sigma$.

- BV master equation on $\Sigma \implies$ integrability.
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- BV master equation on $\Sigma \implies \text{integrability}.$
- The leading effective action is computed by Saito’s primitive form/Barannikov-Kontsevich’s semi-infinite period map, which is the analogue of $e^{u/t}$.
BV-formalism is a general method to quantize gauge theory.

**Definition**

A *differential BV algebra* is a triple \((\mathcal{A}, Q, \Delta)\) where

- \(\mathcal{A}\) is a graded commutative algebra.
- \(Q\) is a derivation such that \(\text{deg}(Q) = 1\), \(Q^2 = 0\).
- \(\Delta : \mathcal{A} \to \mathcal{A}\) is a second order operator such that \(\text{deg}(\Delta) = 1\), \(\Delta^2 = 0\). The failure of being a derivation defines the BV-bracket:

\[
\{ a, b \} = \Delta(ab) - (\Delta a)b \pm a\Delta b,
\]

\(\forall a, b \in \mathcal{A}\).

\(Q\) and \(\Delta\) are compatible:

\(Q\Delta + \Delta Q = 0\).
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- \(Q\) and \(\Delta\) are compatible: \(Q\Delta + \Delta Q = 0\).
Let \((V, Q, \omega)\) be a (-1)-symplectic dg vector space

\[
\omega \in \wedge^2 V^*, \quad Q(\omega) = 0, \quad \deg(\omega) = -1. 
\]

It identifies

\[
V^* \simeq V[1].
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Let \(K = \omega^{-1} \in \text{Sym}^2(V)\) be the Poisson kernel under

\[
\wedge^2 V^* \cong \text{Sym}^2(V)[2]
\]

\[
\begin{array}{c}
\omega \\
K
\end{array}
\]

Let \(\mathcal{O}(V) := \widehat{\text{Sym}}(V^*) = \prod_n \text{Sym}^n(V^*).\) Then \((\mathcal{O}(V), Q)\) is a commutative dga.
The degree 1 Poisson kernel $K$ defines a BV operator

$$\Delta_K : \mathcal{O}(V) \to \mathcal{O}(V) \quad \text{by}$$

$$\Delta_K(\varphi_1 \cdots \varphi_n) = \sum_{i,j} \pm (K, \varphi_i \otimes \varphi_j) \varphi_1 \cdots \hat{\varphi_i} \cdots \hat{\varphi_j} \cdots \varphi_n, \quad \varphi_i \in V^*.$$

Then $(\mathcal{O}(V), Q, \Delta_K)$ defines a differential BV algebra. We have

$$(−1)\text{-shifted dg symplectic} \implies \text{differential BV}.$$
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Remark: this process is well-defined for Poisson instead of symplectic. In fact, as we will see, the Poisson kernel for topological B-model is degenerate.
Let \((\mathcal{A}, Q, \Delta)\) be differential BV. Let \(I = I_0 + I_1 \hbar + \cdots \in \mathcal{A}[[\hbar]]\).

**Definition**

\(I\) is said to satisfy **quantum BV-master equation (QME)** if

\[
(Q + \hbar \Delta) e^{I/\hbar} = 0.
\]

This is equivalent to

\[
QI + \hbar \Delta I + \frac{1}{2} \{I, I\} = 0.
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The leading \(\hbar\)-order \(I_0\) satisfies

\[
QI_0 + \frac{1}{2}\{I_0, I_0\} = 0
\]

which is called the **classical BV-master equation (CME)**.
Quantum master equation arises as the quantum consistency condition for quantum field theory with gauge symmetries. At the classical level, classical master equation says that

\[ Q + \{l_0, -\} \]

squares zero, which describes the infinitesimal gauge transformations. In mathematical terminology, this defines an \( L_\infty \)-algebra.
QFT case

QFT deals with infinite dimensional geometry. Typically the toy model \((V, Q, \omega)\) is modified to \((\mathcal{E}, Q, \omega)\) as follows:

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The serious problem (UV-divergence) is that

$$\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

is **ill-defined** since we cannot pair two distributions. *Renormalization* is required!
The basic idea is

$$H^\ast(distribution, Q) = H^\ast(smooth, Q).$$

Therefore we can replace $K_0$ by something smooth, and remember the original theory in a homotopic way.
Since $Q(K_0) = 0$, we can find $P_r$ such that

$$K_0 = K_r + Q(P_r)$$

and $K_r$ is smooth. Therefore $\Delta_{K_r} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$ is well-defined.
Effective BV-formalism

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- Different choices of $K_r$ leads to homotopic equivalent structures. The connecting homotopy will be called

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- BV master equation is formulated homotopically.
Effective BV quantization

We say the theory is UV finite if \( \lim_{r \to 0} BV[r] \) exists.

\[ r = 0 \text{ (unrenormalized)} \]
Effective BV quantization

Definition

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Some examples of free CFT in 2d

- free boson: $\int \partial \phi \wedge \bar{\partial} \phi$.
- bc-system: $\int b \wedge \bar{\partial} c$.
- $\beta\gamma$-system: $\int \beta \wedge \bar{\partial} \gamma$. 

We will study effective BV quantization in 2d for chiral deformation of free CFT's of the form:

$$S = \text{free CFT's} + I$$

where $I = \int d^2z L_{\text{hol}}(\partial_z \phi, b, c, \beta, \gamma)$.
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$$I = \int d^2 z L^{\text{hol}}(\partial_z \phi, b, c, \beta, \gamma)$$

where $L^{\text{hol}}$ is a lagrangian density involving only holomorphic derivatives of $\partial_z \phi, b, c, \beta, \gamma$. 
Vertex algebras

A vertex algebra is a vector space $\mathcal{V}$ with structures

- state-field correspondence

\[
\mathcal{V} \rightarrow \text{End}(\mathcal{V})[[z, z^{-1}]] \\
A \rightarrow A(z) = \sum_{n} A_{(n)} z^{-n-1}
\]

- vacuum: $|0\rangle \rightarrow 1$.

- translation operator, locality, etc.
We can define OPE’s of fields by

\[ A(z)B(w) = \sum_{n \in \mathbb{Z}} \frac{(A(n) \cdot B)(w)}{(z - w)^{n+1}} \]

or simply the singular part

\[ A(z)B(w) \sim \sum_{n \geq 0} \frac{(A(n) \cdot B)(w)}{(z - w)^{n+1}} \]
Vertex algebra $\mathcal{V} \Rightarrow$ Lie algebra $\oint \mathcal{V}$.

As a vector space, the Lie algebra $\oint \mathcal{V}$ has a basis given by $A(k)$'s

$$\oint \mathcal{V} := \text{Span}_\mathbb{C} \left\{ \oint dz z^k A(z) := A(k) \right\}_{A \in \mathcal{V}, k \in \mathbb{Z}}.$$

The Lie bracket is determined by the OPE

$$[A(m), B(n)] = \sum_{j \geq 0} \binom{m}{j} (A(j) B)_{m+n-j}.$$
Let $\mathfrak{h}$ be a graded vector space with $\text{deg} = 0$ symplectic pairing

$$\langle -, - \rangle : \wedge^2 \mathfrak{h} \to \mathbb{C}.$$ 

We obtain a vertex algebra structure on the free differential ring

$$\mathcal{V}[\mathfrak{h}] = \mathbb{C}[\partial^k a^i], \quad \{a^i\} \text{ is a basis of } \mathfrak{h}, k \geq 0.$$ 

(or $\mathcal{V}[[\mathfrak{h}]] = \mathbb{C}[[\partial^k a^i]]$).
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(or $\mathcal{V}[[\mathfrak{h}]] = \mathbb{C}[[\partial^k a^i]]$). The OPE’s are generated by

$$a(z)b(w) \sim \left( \frac{i\hbar}{\pi} \right) \frac{\langle a, b \rangle}{z - w}, \quad \forall, \ a, b \in \mathfrak{h}.$$ 

$\Rightarrow \mathcal{V}[\mathfrak{h}]$ is a combination of $bc$ and $\beta\gamma$ systems.
We consider QFT on $\Sigma$ where

$$\Sigma = (\mathbb{C}, z), \quad (\mathbb{C}^*, e^{2\pi i z}), \quad \text{or} \quad (E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}_\tau), z)$$

with volume form $dz$. Let $\mathfrak{h}$ be a graded symplectic space as above. We obtain the following BV triple $(\mathcal{E}, Q, \omega)$.
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3. (-1)-symplectic pairing

$$\omega(\varphi_1, \varphi_2) := \int dz \wedge \langle \varphi_1, \varphi_2 \rangle, \quad \varphi_i \in \mathcal{E}.$$ 

$\Rightarrow$ effective BV formalism
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$$\int V[[h^\vee]] \to \text{chiral local functional on } \mathcal{E}.$$
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Given

$$I = \sum \int \partial^{k_1} a_1 \cdots \partial^{k_n} a_n \in \mathcal{V}[[h^\vee]], \quad \text{where } a_i \in h^\vee,$$

it is mapped to

$$\hat{I}[\varphi] := \sum \int dz z^{\partial^{k_1} a_1(\varphi)} \cdots z^{\partial^{k_n} a_n(\varphi)}$$

for $\varphi \in \mathcal{E} = \Omega^0,^* (\Sigma) \otimes h$. 

Lemma

The triple $(\int \mathcal{V}[[h^\vee]], \delta, [-, -])$ defines a dgLa.
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**Lemma**

The triple $(\int \mathcal{V}[[h^\vee]], \delta, \{-, -\})$ defines a dgLa.
We consider the following 2d chiral QFT

\[ S = \text{free CFT} + \hat{I}, \quad I \in \oint \mathcal{V}[h^\vee][[\hbar]]. \]

1. **The theory is UV-finite.**

2. **Solutions of (homotopic) effective BV master equations**

\[ \Leftrightarrow \delta I + \frac{1}{2} \left( \frac{i\hbar}{\pi} \right)^{-1} [I, I] = 0, \quad I \in \oint \mathcal{V}[[h^\vee]][[\hbar]]. \]

In other words, \( I \) is a MC element of the dgLa \( \oint \mathcal{V}[[h^\vee]][[\hbar]]. \)

3. **The generating functions are almost holomorphic modular forms, i.e., modular of the form**

\[ \sum_{k=0}^{N} \frac{f_k(\tau)}{(\text{im } \tau)^k}. \]
Consider the following chiral deformation of free boson on the elliptic curve $E_\tau$

$$S = \frac{1}{2} \int_{E_\tau} \partial \phi \wedge \bar{\partial} \phi + \frac{1}{3!} \int_{E_\tau} \frac{d^2z}{\text{im} \tau} (\partial_z \phi)^3.$$

Let’s look at a two-loop diagram

Here the propagator is $P(z; \tau) = \mathcal{P}(z, \tau) + \frac{\pi^2}{3} E_2^*$ where

$$\mathcal{P} \text{ Weierstrass P-function, } E_2^* = E_2 - \frac{3}{\pi} \frac{1}{\text{im} \tau}.$$


Naively $\mathcal{P}$ has a second order pole and $\int_{E_r} \mathcal{P}^3$ would be divergent.

However, in the sense of homotopic renormalization, its renormalized value has a well-defined limit $r \to 0$, whose value can be computed as follows:
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$$0 \to \mathcal{C} \xrightarrow{\partial} \mathcal{M} \xrightarrow{\partial} \Omega^{\text{II}} \to 0$$

where $\mathcal{M}$ is the sheaf of meromorphic functions, and $\Omega^{\text{II}}$ is the sheaf of abelian differentials of second kind.
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where $\mathcal{M}$ is the sheaf of meromorphic functions, and $\Omega^{II}$ is the sheaf of abelian differentials of second kind. Then

$$\left[p^3 \frac{dz}{\text{im} \tau} \wedge d\bar{z}\right] \in H^1(E_\tau, \Omega^{II}) \to H^2(E_\tau, \mathbb{C}) \xrightarrow{\int} \mathbb{C}.$$ 

represents the renormalized integral.
We find the following expression

\[
\frac{1}{\pi^6} \int_{E_\tau} \frac{d^2z}{\text{im } \tau} \mathbf{P}^3 = \frac{2^2}{3^3 5} E_6 + \frac{2}{3^2 5} E_4 E_2^* - \frac{2}{3^3} (E_2^*)^3
\]
We find the following expression

\[
\frac{1}{\pi^6} \int_{E_\tau} \frac{d^2z}{\text{im } \tau} P^3 = \frac{2^2}{3^3 5} E_6 + \frac{2}{3^2 5} E_4 E_2^* - \frac{2}{3^3} (E_2^*)^3
\]

Under the \( \bar{\tau} \rightarrow \infty \) limit, which amounts to replace \( E_2^* \rightarrow E_2 \),

\[
\Rightarrow \frac{1}{\pi^6} \oint_A dz P^3 = \frac{2^2}{3^3 5} E_6 + \frac{2}{3^2 5} E_4 E_2 - \frac{2}{3^3} (E_2)^3.
\]

reducing to the A-cycle integral as computed by M.Douglas.
Let $X$ be a CY. The field content of BCOV theory (in the generalized sense of [Costello-L]) is given by the complex

$$\mathcal{E} = \text{PV}(X)[[t]], \quad Q = \bar{\partial} + t\partial$$

where $\text{PV}(X) = \Omega^{0,*}(X, \wedge^* T_X)$ and $\partial$ is the divergence operator w.r.t. the CY volume form.
BCOV theory on elliptic curve

We specialize to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$

$$\mathcal{E} = \Omega^{0,*}(E_\tau)[[t]] \oplus \Omega^{0,*}(E_\tau, T_{E_\tau}[1])[t]$$
$$= \Omega^{0,*}(E_\tau) \otimes h.$$ 

where $h = \mathbb{C}[[t, \theta]]$, $\deg(t) = 0$, $\deg(\theta) = -1$. 
We specialize to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}_\tau)$

\[ E \subseteq \Omega^0,(E_\tau)[[t]] \oplus \Omega^0,(E_\tau, T_{E_\tau}[1])[t]] \]

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- $Q = \bar{\partial} + t\partial$, where $\partial = \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \theta}$. 
We specialize to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}_\tau)$

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where $h = \mathbb{C}[[t, \theta]], \deg(t) = 0, \deg(\theta) = -1$. Via our 2d set-up,

- $Q = \bar{\partial} + t\delta$, where $\delta = \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \theta}$.
- The Poisson kernel is degenerate. If we represent $\varphi \in \mathcal{E}$ by

$$
\varphi = \sum_{k \geq 0} b_k t^k + \eta_k \theta t^k,
$$

then the OPE's are generated by

$$
b_0(z)b_0(w) \sim \frac{1}{(z - w)^2}, \quad \text{others } \sim 0.
$$

$b_0$ is dynamical, while $b_{>0}, \eta_\bullet$ are background fields.
Quantum B-model on elliptic curves

Theorem (L)

There exists a canonical solution of quantum master equation for BCOV theory on elliptic curves.

This is proved by analyzing the deformation obstruction complex of the dgLa for the relevant vertex algebra under the boson-fermion correspondence.
Since the theory is UV-finite, we can express the solution of homotopic BV master equation via quantum corrected local functions. We give some explicit description in the so-called stationary sector (which amounts to freeze the background fields):

\[ b_{>0} = 0, \quad \eta \cdot = \text{constants.} \]
Since the theory is UV-finite, we can express the solution of homotopic BV master equation via quantum corrected local functions. We give some explicit description in the so-called \textit{stationary sector} (which amounts to freeze the background fields):

\[ b_{>0} = 0, \quad \eta_\bullet = \text{constants}. \]

The quantum corrected action in the stationary sector is

\[ S = \int \partial \phi \wedge \bar{\partial} \phi + \sum_{k \geq 0} \int \eta_k \frac{W^{(k+2)}(b_0)}{k + 2}, \quad b_0 = \partial_z \phi. \]

where

\[ W^{(k)}(b_0) = \sum_{\sum_i k_i = k \prod_i k_i!} \frac{k_i!}{\prod_i k_i!} \left( \prod_i \frac{1}{i!} (\sqrt{\hbar} \partial_z)^{i-1} b_0 \right)^{k_i} = b_0^k + O(\hbar). \]

are the bosonic realization of the \( W_{1+\infty} \)-algebra.
In the stationary sector, the BV quantum master equation is equivalent to

\[
\left[ \oint W^{(k)}, \oint W^{(m)} \right] = 0, \quad \forall k, m \geq 0,
\]

representing \( \infty \) many commutating vertex operators.

Its classical limit is

\[
\left\{ \oint b_0^k, \oint b_0^m \right\} = 0, \quad \forall k, m \geq 0,
\]

for the Poisson bracket \( \{ b_0(z), b_0(w) \} = \partial_z \delta(z - w) \) that we observe in the beginning.
The generating functions of the quantum BCOV theory are almost holomorphic modular forms. The $\bar{\tau}$-dependence is the famous holomorphic anomaly.

In the stationary sector, the $\bar{\tau} \to \infty$ limit of the generating function can be computed by

$$\text{Tr } q^{L_0} - \frac{1}{24} e^{\frac{1}{\hbar} \sum_{k \geq 0} \oint A \eta_k W^{(k+2)}_{k+2}}$$

which coincides with the stationary GW-invariants on the mirror elliptic curve computed by Okounkov-Pandharipande,

$$\Rightarrow \text{higher genus mirror symmetry.}$$

This generalizes the work of Dijkgraaf on the cubic interaction.
Remarks

One way to understand the interaction of B-model on $E$ is via

$$pt \times E \to E.$$  

In general, we consider $X \times E \to E$, whose compactification along $X$ gives rise to an effective 2d chiral theory on $E$. Then similarly we will find $\infty$ commutating Hamiltonians, which turns out to be Dubrovin-Zhang’s Principal integrable hierarchy (in progress with Weiqiang He and Philsang Yoo).
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- Landau-Ginzburg twisting. Classical BCOV theory is equivalent to Saito’s theory of primitive forms. The classical commuting hamiltonians can be computed by replacing $e^{u/t}$ with the so-called primitive form.
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- Couple BCOV theory with Witten’s HCS [Costello-L, 2016].
Thank You!