Multivariate trace inequalities

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What are trace inequalities?

Trace inequalities relate traces of various products of Hermitian matrices.

Typically, they become equalities for commuting matrices. Roughly speaking, they allow to control non-commutative terms “on average” (i.e., inside the trace).

Theorem (Golden and Thompson ’65)

Let \( A, B \in \mathbb{C}^{m \times m} \) be Hermitian matrices. Then

\[
\text{Tr}[e^{A+B}] \leq \text{Tr}[e^A e^B]
\]

- Compare this simple form to the elaborate expansion provided by the Baker-Campbell-Hausdorff formula.
- Lots of applications: Statistical mechanics; quantum information theory; random matrix theory.
- Follows from the Cauchy-Schwarz inequality.
Lieb’s three-matrix inequality

It is not obvious how to generalize $\text{Tr}[e^{A+B}] \leq \text{Tr}[e^Ae^B]$ to more than two matrices. Naive generalizations are false:

$$\text{Tr}[e^{A+B+C}] \not\leq \begin{cases} \text{Tr}[e^Ae^Be^C], \\ \text{Tr}[e^{A+B/2}e^{C}e^{B/2}]. \end{cases}$$

But:

**Theorem (Lieb ’76)**

Let $A, B, C \in \mathbb{C}^{m \times m}$ be Hermitian matrices. Then

$$\text{Tr}[e^{A+B+C}] \leq \int_0^\infty \text{Tr} \left[ e^A \frac{1}{e^{-B} + \tau I} e^C \frac{1}{e^{-B} + \tau I} \right] d\tau$$

- The strange expression

$$\int_0^\infty \frac{1}{X + \tau I} Y \frac{1}{X + \tau I} d\tau = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \log (X + \epsilon Y)$$

is a non-commutative analogue of $X^{-1}Y$.
- Proof uses convex matrix functions (Löwner’s theorem helps).
A recent breakthrough

An extension of Lieb’s three-matrix inequality to $n \geq 4$ matrices was missing, until last year.

**Theorem (Sutter, Berta and Tomamichel 2016)**

For $n \geq 2$, let $A_1, \ldots, A_n \in \mathbb{C}^{m \times m}$ be Hermitian matrices. Then

$$\text{Tr} \left[ \exp \left( \sum_{k=1}^{n} A_k \right) \right] \leq \int_{-\infty}^{\infty} \text{Tr} \left[ e^{A_n} e^{\frac{1+it}{2} A_{n-1}} \ldots e^{\frac{1+it}{2} A_2} e^{A_1} e^{\frac{1-it}{2} A_2} \ldots e^{\frac{1-it}{2} A_{n-1}} \right] \beta(t) \, dt$$

where $\beta(t) := \frac{\pi}{2} \left( 1 + \cosh(\pi t) \right)^{-1}$ is an explicit probability density.

- For $n = 3$, this is actually Lieb’s inequality in disguise.
- Their new $n = 4$ inequality is useful in quantum information theory.
- Proof uses complex interpolation (Stein-Hirschmann in Schatten spaces).
Q: Can the SBT inequalities be formulated in Lieb's form (i.e., in terms of resolvents $\frac{1}{X+\tau I}$) for $n > 3$?

This is not just an academic question: The unitaries $e^{it\frac{1}{2}A_k}$ appear to block other applications of the new SBT inequalities, e.g., to random matrix theory (large deviations; bounds on the Lyapunov exponent). If the answer is yes, then we can use the “resolvent formalism” to remove the unitaries up to explicit commutators.

**Theorem (arXiv:1708.04836)**

A: Yes. E.g., for real symmetric matrices $A_1, A_2, A_3, A_4$,

$$\text{Tr}_\mathcal{H}[e^{A_1+A_2+A_3+A_4}] \leq \int_0^\infty \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} \left[ P \frac{1}{e^{-A_2} \otimes e^{-A_3 + \tau I}} (e^{A_1} \otimes e^{A_4}) \frac{1}{e^{-A_2} \otimes e^{-A_3 + \tau I}} \right] d\tau$$
Thank you for your attention!