Lifting Galois representations

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Cohomology with $\mathbb{C}$-coefficients

Let $X$ be a smooth projective variety over $\mathbb{Q}$. The Hodge filtration gives $H^i_{\text{sing}}(X,\mathbb{C}) \cong \bigoplus_{p+q=i} H^p(X,\Omega^q)$.

Let $h_{p,q} = \dim \mathbb{C}H^p(X,\Omega^q)$.

(Riemann) If $X$ is an abelian variety of dimension $g$, then $H^1(X,\mathcal{O})$ and $H^0(X,\Omega^1)$ are $g$-dimensional and the Hodge filtration (with rational structure) determines $X$ up to isogeny.
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Cohomology with \( \mathbb{Q}_p \)-coefficients

Let \( H^i = H^i_{\text{et}}(X/\mathbb{Q}, \mathbb{Q}_p) \). \( G_{\mathbb{Q}_p} \) acts on \( H^i \).

Grothendieck–Lefschetz:
\[
\#X(F_\ell) = \sum_{i} (-1)^i \text{Tr}(\text{Frob}_\ell, H^i) \text{ for almost all } \ell.
\]

Let \( Z_p(1) \) def = Hom(\( \mathbb{Q}_p/\mathbb{Z}_p \), \( \mu_{\mathbb{Q}_p} \)).

Faltings:
\[
H^i \otimes \mathbb{Q}_p \mathbb{C}_p \cong \bigoplus_{p+q=i} H^p(X/\mathbb{Q}_p, \Omega^q) \otimes \mathbb{Q}_p \mathbb{C}_p(-q)
\]
This is \( G_{\mathbb{Q}_p} \)-equivariant, so \( h^p,q \) are determined.

\( \lambda \) (= HT weights of \( H^i \)) is the multiset with \( -q \) appearing \( h^p,q \) times.

\( X \) has good reduction at \( p \) \( \Rightarrow \) \( H^i \) is crystalline.

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If $X$ is an abelian variety, then $H^1$ determines $X$ up to isogeny.
Galois deformations

Let $\rho: G_{\mathbb{Q}^p} \rightarrow \text{GL}_n(F_p)$. Let $X_{\lambda}(\rho)$ be \{\rho: $G_{\mathbb{Q}^p} \rightarrow \text{GL}_n(Q_p)$ crystalline of HT weights $\lambda$ | $\rho \equiv \rho \mod m$\}. How many connected components does $X_{\lambda}(\rho)$ have? Is $X_{\lambda}(\rho)$ nonempty?

Let $\rho \sim F_p(a) \sim \mu \otimes a_p$. Note that $F_p(a) \sim F_p(b) \iff a \equiv b \mod p-1$. Then $\lambda \equiv a \mod p-1 \iff X_{\lambda}(F_p(a)) \neq \emptyset$ (case $\lambda$ is a lift) in which case $X_{\lambda}(F_p(a)) \sim \mathbb{Z}/p$. 

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Definition
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If $n = 1$, the Serre weights are $a$-th powers of $\text{St}_a \sim \text{St}_b \iff a \equiv b \mod p - 1$. 

$X_\lambda(\mathbb{F}_p(a)) \neq \emptyset \iff \text{St}_\lambda \sim \text{St}_a$. 

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\[ X^\lambda(\mathbb{F}_p(a)) \neq \emptyset \iff \text{St}^\lambda \cong \text{St}^a. \]
Compatibility of weights

Assume that $\lambda$ is regular, i.e., $h_p, q \leq 1$ for all $p, q$.

Let $\eta = (n-1, n-2, \ldots, 1, 0)$.

$\lambda \Rightarrow \text{alg. rep. } V(\lambda - \eta)$ of $\text{GL}_n$.

Conjecture

Let $\rho: G \to \text{GL}_n(\mathbb{F}_p)$.

$\exists$ a set of Serre weights $W(\rho)$ such that $X(\lambda)(\rho) \neq \emptyset \iff W(\rho) \cap JH(V(\lambda - \eta)(\mathbb{F}_p)) \neq \emptyset$.

The conjecture holds for $n = 1$: If $\rho = \mathbb{F}_p(a)$, then $W(\rho) = \text{St}_a$.

The conjecture holds for $n = 2$ using the $p$-adic Langlands correspondence of Colmez.
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Partial results in higher dimensions

Theorem (L., Le Hung, Levin, Morra)

If $n = 3$ and $\rho$ is generic, $\exists W(\rho)$ such that $W(\rho) \cap JH(V(\lambda - \eta)(Fp)) \neq \emptyset \Rightarrow X(\lambda)(\rho) \neq \emptyset$.

Moreover, the converse holds in the potentially diagonalizable case and in the tamely potentially crystalline case when $\lambda = \eta$.

Theorem (L., Le Hung, Levin)

Let $\rho$ be semisimple and generic ($n$ is arbitrary). Then $\exists W^? (\rho)$ such that the conjecture (with $W^? (\rho)$ replacing $W(\rho)$) holds in the tamely potentially crystalline case when $\lambda = \eta$.

What if $\lambda$ is not regular?
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Let \( \bar{\rho} \) be semisimple and generic (\( n \) is arbitrary). Then \( \exists W?(\bar{\rho}) \) such that the conjecture (with \( W?(\bar{\rho}) \) replacing \( W(\bar{\rho}) \)) holds in the tamely potentially crystalline case when \( \lambda = \eta \).
Partial results in higher dimensions

Theorem (L., Le Hung, Levin, Morra)
If $n = 3$ and $\bar{\rho}$ is generic, $\exists W(\bar{\rho})$ such that $W(\bar{\rho}) \cap JH(V(\lambda - \eta)(\mathbb{F}_p)) \neq \emptyset \implies X^\lambda(\bar{\rho}) \neq \emptyset$. Moreover, the converse holds in the potentially diagonalizable case and in the tamely potentially crystalline case when $\lambda = \eta$.

Theorem (L., Le Hung, Levin)
Let $\bar{\rho}$ be semisimple and generic ($n$ is arbitrary). Then $\exists W^?(\bar{\rho})$ such that the conjecture (with $W^?(\bar{\rho})$ replacing $W(\bar{\rho})$) holds in the tamely potentially crystalline case when $\lambda = \eta$.

What if $\lambda$ is not regular?