Arithmetic theta series

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The theta correspondence between automorphic representations of groups \((H, G)\) in a reductive dual pair, as well as its local version, has proved to be a useful tool in the study of such representations.

The basis for this correspondence is the use of theta functions \(\theta(h, g; \varphi)\) built from Schwartz functions \(\varphi\), say in some Schrödinger model of the Weil representation, as integral kernels to transport cuspidal automorphic functions from one group to the other.

The seesaw identities, Siegel-Weil formula, and the doubling method then yield criteria for the non-vanishing of such theta lifts in terms of special values of L-functions and local obstructions.
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The seesaw identities, Siegel-Weil formula, and the doubling method then yield criteria for the non-vanishing of such theta lifts in terms of special values of \(L\)-functions and local obstructions.
As a variant of this, when the group \( G \) is a classical group \( O(p, q) \), \( U(p, q) \) or \( Sp(p, q) \), Millson and I constructed theta functions valued in the deRham complex for the associated locally symmetric manifold, \( M = \Gamma \backslash D \).

These ‘geometric’ theta series are closely linked to a certain type of locally symmetric cycles in \( M \).

The geometric theta series are closed as differential forms and, passing to cohomology, they give rise to a theta correspondence between automorphic forms on \( H \) and cohomology classes on \( M \).

Such correspondences are the starting point for many applications, among them the recent striking results of Bergeron-Millson-Moeglin (2014).

Among other things, they establish new cases of the Hodge conjecture for certain ball quotients where \( M \) is a quasi-projective variety and the locally symmetric cycles are, in fact, algebraic cycles on \( M \).
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For a long time now, I have been pursuing the notion that there should be another ‘generation’ of theta series, the ‘arithmetic’ theta series. Very roughly, these arithmetic theta series should arise in the case where $M$ is a Shimura variety with a regular integral model $\mathcal{M}$. The idea is to constructed generating series for classes of certain ‘special cycles’ in $\mathcal{M}$ in the arithmetic Chow groups $\widehat{CH}^r(\mathcal{M})$. The goal is then to show that these series define $\widehat{CH}^r(\mathcal{M})$-valued automorphic forms on $H$. These arithmetic theta series would then provide an arithmetic theta correspondence between automorphic forms on $H$ and classes in the arithmetic Chow groups. An arithmetic Siegel-Weil formula would then provide a criterion for the nonvanishing of arithmetic theta lifts in term of values of derivatives of $L$-functions and local obstructions.
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With this setting as background and motivation, I want to report on the recent results of a joint project\(^1\) with Jan Bruinier, Ben Howard, Michael Rapoport and Tonghai Yang in which we construct arithmetic theta series valued in \(\widehat{\text{CH}}^1(\mathcal{M})\) in the case\(^2\) \(G = U(n - 1, 1)\) for \(H = U(1, 1)\). The references are:

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Here is some notation:

\( \mathcal{K} = \) imaginary quadratic field with odd discr. \(-D\)

\( \mathcal{W} = \) hermitian space over \( \mathcal{K} \) of signature \( (n - 1, 1) \)

\( \mathcal{W}_0 = \) hermitian space over \( \mathcal{K} \) of signature \( (1, 0) \)

\( G = \{(g_0, g) \in GU(\mathcal{W}_0) \times GU(\mathcal{W}) \mid \nu(g_0) = \nu(g)\} \)

\( \alpha, \alpha_0 = \) self-dual \( O_{\mathcal{K}} \)-lattices in \( \mathcal{W} \) and \( \mathcal{W}_0 \)

\( K = G(\mathbb{A}_f) \cap (K_{\alpha_0} \times K_{\alpha}) \), compact open

\( \text{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K \), the Shimura variety.

The presence of the ‘extra’ factor coming from \( \mathcal{W}_0 \) is essential in the definition of the special cycles.
The unitary Shimura variety

Here is some notation:

\( k = \) imaginary quadratic field with odd discr. \(-D\)
\( W = \) hermitian space over \( k \) of signature \((n - 1, 1)\)
\( W_0 = \) hermitian space over \( k \) of signature \((1, 0)\)
\( G = \{ (g_0, g) \in \text{GU}(W_0) \times \text{GU}(W) \mid \nu(g_0) = \nu(g) \} \)
\( a, a_0 = \) self-dual \( O_k \)-lattices in \( W \) and \( W_0 \)
\( K = G(\mathbb{A}_f) \cap (K_{a_0} \times K_a), \) compact open
\( \text{Sh}(G, \mathcal{D})(\mathbb{C}) = \frac{G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)}{K}, \) the Shimura variety.

The presence of the ‘extra’ factor coming from \( W_0 \) is essential in the definition of the special cycles.
A modular interpretation

To define the integral model, consider the following moduli problem:

To an $O_k$-scheme $S$ assign the groupoid of triples $(A, \iota, \psi)$ where

- $A \rightarrow S$ an abelian scheme of relative dim. $n$
- $\iota : O_k \rightarrow \text{End}(A)$ an $O_k$ action such that
  \[
  \det(T - \iota(\alpha)|\text{Lie}(A)) = (T - \alpha)^{n-1}(T - \bar{\alpha}) \in O_S[T],
  \]
- $\psi : A \rightarrow A^\vee$ a principal polarization such that
  \[
  \iota(\alpha)^\dagger = \iota(\bar{\alpha}). \quad \dagger = \text{Rosati for } \psi.
  \]

This a not quite good enough at primes dividing the discriminant of $k$.

To obtain a better model, enhance the data to $(A, \iota, \psi, F_A)$, where

- $F_A \subset \text{Lie}(A) = O_k$-stable, locally direct $O_S$ submodule of rank $n - 1$

with $O_k$-acting on $F_A$ via $O_k \rightarrow O_S$, and
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with $O_k$-acting on $F_A$ via $O_k \to O_S$, and with $O_k$-acting on $\text{Lie}(A)/F_A$ via the conjugate.
The resulting moduli stack\textsuperscript{3} \( \mathcal{M}_{n-1,1}^{\text{Kra}} \) over \( \text{Spec} \, O_k \) is regular and flat.

The moduli stack \( \mathcal{M}_{1,0} \) over \( \text{Spec} \, O_k \) defined via triples \((A_0, \iota_0, \psi_0)\) as above is already smooth\textsuperscript{4} over \( \text{Spec} \, O_k \).

If we denote the generic fibers of these stacks by \( \mathcal{M}_{n-1,1} \) and \( \mathcal{M}_{1,0} \), then

\[
\text{Sh}(G, D) \subset \mathcal{M}_{1,0} \times_k \mathcal{M}_{n-1,1}
\]

is an open and closed substack, characterized by the existence of an isomorphism, for every prime \( \ell \),

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\text{Hom}_{O_k}(T_\ell A_0, s, T_\ell A_s) \simeq \text{Hom}_{O_k}(a_0, a) \otimes \mathbb{Z}_\ell
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at every geometric point \( s \). This is required to be an isometry for the natural \((O_k)_\ell\)-hermitian form on the two sides.

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\textsuperscript{3}the Krämer model

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We obtain our integral model by taking\(^5\) the Zariski closure

\[
\begin{align*}
\text{Exc} & \rightarrow S_{\text{Kra}} & \rightarrow \mathcal{M}_{(1,0)} \times \mathcal{M}_{(n-1,1)}^{\text{Kra}} \\
\downarrow & & \downarrow \\
\text{Sing} & \rightarrow S_{\text{Pap}} & \rightarrow \mathcal{M}_{(1,0)} \times \mathcal{M}_{(n-1,1)}^{\text{Pap}} \\
\uparrow & & \uparrow \\
\text{Sh}(G, \mathcal{D}) & \rightarrow \mathcal{M}_{(1,0)} \times \mathcal{M}_{(n-1,1)} \\
\text{Zariski closure} & & \text{blowup}
\end{align*}
\]

Here \(\mathcal{M}_{(n-1,1)}^{\text{Pap}}\) is an intermediate model defined by adding the Pappas wedge condition rather than the Krämer condition. It has isolated singular points in fibers over ramified primes. These are blown up to an exceptional divisor in the Krämer model.

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\downarrow & & \downarrow & & \downarrow & \\
\text{Sing} & \rightarrow & S_{\text{Pap}} & \rightarrow & M_{(1,0)} \times M_{(n-1,1)}^{\text{Pap}} & \leftarrow \text{Zariski closure} \\
\uparrow & & \uparrow & & \uparrow & \\
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\(^5\text{Not quite...}\)
The upshot of the previous discussion is that we have nice integral models

\[ S_{\text{Kra}} \longrightarrow S_{\text{Pap}}, \]

related by a blowup.

They both have nice toroidal compactifications

\[ S_{\text{Kra}}^* \longrightarrow S_{\text{Pap}}^*, \]

with boundary divisors to be discussed in a moment.

In a careful treatment, the two must be carried along since:

- \( S_{\text{Pap}}^* \) is not regular but every vertical Weil divisor meets the boundary.
- \( S_{\text{Kra}}^* \) is regular but \( \text{Exc} \) does not meet the boundary.

For the moment, we write \( S = S_{\text{Kra}} \subset S_{\text{Kra}}^* = S^* \).
Special divisors in the integral models

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Special divisors in the integral models

The upshot of the previous discussion is that we have nice integral models

\[ S_{\text{Kra}} \rightarrow S_{\text{Pap}}, \]

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For an $O_k$-scheme $S$, an $S$-valued point of $S$ corresponds to a pair $(A_0, A)$ of principally polarized abelian schemes with $O_k$-action and some additional equipment.

For such a pair, the $O_k$-lattice

$$L(A_0, A) = \text{Hom}_{O_k}(A_0, A)$$

has a natural positive definite hermitian form defined by

$$\langle x_1, x_2 \rangle = x_2^\vee \circ x_1 \in \text{End}_{O_k}(A_0) \sim \to O_k,$$

$$A_0 \xrightarrow{x_1} A \xleftarrow{x_2^\vee} A^\vee.$$ 

In some sense, the arithmetic theta series is attached to this family of hermitian spaces.
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where $\psi_0^{-1}$ and $\psi$ are bijections of the lattices $A_0$ and $A$.

In some sense, the arithmetic theta series is attached to this family of hermitian spaces.
For $m \in \mathbb{Z}_{>0}$, let

$$
\mathcal{Z}(m)(S) = \left( \begin{array}{c}
\text{groupoid of triples } (A_0, A, x), \\
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x \in L(A_0, A), \text{ with } \langle x, x \rangle = m
\end{array} \right) \rightarrow S(S).
$$

The arithmetic special divisors $\mathcal{Z}(m)$’s are Cartier divisors on $S$. Conceptually, they are the loci where the abelian variety $A$ is equipped with an elliptic curve factor $A_0$.

Let $\mathcal{Z}^*(m)$ be the Zariski closure of $\mathcal{Z}(m)$ in $S^*$, the toroidal compactification of $S$. 
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It is now time to say something about the compactification $S^*$.

Recall that our Shimura variety is

$$\text{Sh}(G, D)(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K,$$

where

$D = \text{negative lines in } V_{\mathbb{R}},$

and

$$V = \text{Hom}_k(W_0, W), \quad \langle x, y \rangle = y^\vee \circ x.$$

Note that the definition of $V$, where the extra hermitian space $W_0$ seems unnecessary, is motivated by the definition of the special cycles where the role of the ‘auxiliary’ elliptic curve $A_0$ is essential.

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Stephen Kudla (Toronto)

Arithmetic theta series
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The rational boundary components (cusps) of $D$ correspond to the isotropic $k$-lines $J \subset V$. 
The cusps of $\text{Sh}(G, \mathcal{D})(\mathbb{C})$ are then indexed by pairs $\Phi = (J, g)$, with $J \subset V$ an isotropic $k$-line and $g \in G(\mathbb{A}_f)$, modulo a suitable equivalence.

Associated to $\Phi$ is a filtration

$$0 \subset J \subset J^\perp \subset V$$

and an integral version

$$0 \subset J \cap gL \subset J^\perp \cap gL \subset gL = \text{Hom}_{O_k}(g\alpha_0, g\alpha).$$

The hermitian lattice

$$L_\Phi := (gL \cap J^\perp)/gL \cap J$$

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The toroidal compactification of \( \text{Sh}(G, \mathcal{D})(\mathbb{C}) \) is then obtained by blowing up the cusps in the minimal compactification

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\text{Sh}(G, \mathcal{D})(\mathbb{C})^{\text{BB}} \hookrightarrow M_{(1,0)}(\mathbb{C}) \times M_{(n-1,0)}^{\text{BB}}
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to the abelian varieties \( B_\Phi(\mathbb{C}) \) of dimension \( n - 2 \), where

\[
B_\Phi = E \otimes_{\mathcal{O}_k} L_\Phi,
\]

for \( E \rightarrow M_{(1,0)} \) the universal CM-elliptic scheme. This picture propagates to the integral model.

For the finite isometry group

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\Delta_\Phi = \mathcal{O}_k^\times \times \text{U}(L_\Phi),
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where the boundary is a smooth divisor, flat over \( O_k \).

We next augment the divisor \( \mathcal{Z}^*(m) \) by adding a rational linear combination of boundary divisors:

\[ \mathcal{Z}^{\text{tot}}(m) := \mathcal{Z}^*(m) + \mathcal{B} \mathcal{D}(m), \]

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The generating series

We are almost ready to complete the definition of the arithmetic theta series

\[ \hat{\phi}(\tau) = \hat{Z}^{\text{tot}}(0) + \sum_{m=1}^{\infty} \hat{Z}^{\text{tot}}(m) q^m \in \hat{\text{CH}}^1(Q(S^*)). \]

The remaining issues are:

(1) The addition of the Green functions needed to define classes in the arithmetic Chow group \( \hat{\text{CH}}^1_Q(S^*) \), and

(2) the definition of the constant term \( \hat{Z}^{\text{tot}}(0) \).

Here recall that classes in \( \hat{\text{CH}}^1_Q(S^*) \) are given by pairs \( (\mathcal{Z}, g_{\mathcal{Z}}) \) where \( \mathcal{Z} \) is a divisor on \( S^* \) and \( g_{\mathcal{Z}} \) is a Green function on \( S^*(\mathbb{C}) \setminus \mathcal{Z}(\mathbb{C}) \) with

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To define $\mathcal{E}(0)$, note that

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$$-\frac{1}{4\pi}e^{-\gamma} \langle \cdot, \cdot \rangle_V,$$

where $\gamma = -\Gamma'(1)$ is Euler’s constant$^6$.

A natural extension of (the inverse of) $\omega_{an}$ to $S = S_{\text{Kra}}$ is defined by

$$\omega^{-1} = \text{Lie} (A_0) \otimes \text{Lie} (A)/\mathcal{F}_A.$$

Moreover, there is a distinguished extension of $\omega$ to $S^*$ uniquely determined by certain data at the boundary.

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Finally, the metric on $\omega$ extends to a metric with log – log singularities along the boundary\(^7\) of $S^*(\mathbb{C}) \setminus S(\mathbb{C})$, so we obtain a class\(^8\)

$$\hat{\omega} = (\omega, \| \cdot \|) \in \widehat{\text{Pic}}_Q(S^*) \simeq \widehat{\text{CH}}^1_Q(S^*).$$

Define

$$\hat{Z}^{\text{tot}}(0) := \hat{\omega}^{-1} + (\text{Exc}, -\log D) \in \widehat{\text{CH}}^1_Q(S^*).$$

Also, for $m > 0$, define

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\(^7\)Brinier-Howard-Yang (2015)  
\(^8\)in the Burgos-Kramer-Kühn extended arithmetic Chow group
Finally, the metric on $\omega$ extends to a metric with log – log singularities along the boundary\(^7\) of $S^*(\mathbb{C}) \setminus S(\mathbb{C})$, so we obtain a class\(^8\)

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The main theorem

Aside from the detailed definition of the Green functions, we have in hand the complete definition of the arithmetic theta series

$$\hat{\phi}(\tau) = \hat{Z}^{\text{tot}}(0) + \sum_{m=1}^{\infty} \hat{Z}^{\text{tot}}(m) q^m \in \hat{\text{CH}}_{Q}(S^{*})[[q]].$$

**Main Theorem** (BHKRY). The formal series $\hat{\phi}(\tau)$ is a $\hat{\text{CH}}_{Q}(S^{*})$-valued modular form of weight $n$, level $D$, and character $\chi = \chi_{k}^{n}$.

This means that for any $\mathbb{Q}$-linear function $\alpha : \hat{\text{CH}}_{Q}(S^{*}) \rightarrow \mathbb{C}$, the series

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As a formal consequence we have:

**Corollary.** The dimension of the subspace of $\hat{\text{CH}}_{\mathbb{Q}}^1(S^*)$ spanned by the classes $\hat{\mathcal{Z}}^{\text{tot}}(m)$, for $m \geq 0$ is at most $\dim M_n(D, \chi^m_k)$.

Another important consequence is that we can define an *arithmetical theta lift*

$$
\hat{\theta} : S_n(\Gamma_0(D), \chi^m_k) \rightarrow \hat{\text{CH}}^1(S^*), \quad f \mapsto \hat{\theta}(f) = \langle \hat{\phi}, f \rangle_{\text{Pet}}.
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Various constructions analogous to the more familiar ones involving the classical theta correspondence, seesaw identities, etc., give rise to expressions relating height pairings to special values.

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I want to give a sketch of what goes into the proof of the main theorem.
The basic idea is to use the ‘duality method’ introduced by Borcherds.

**Modularity criterion.** For \( k \geq 2 \), and for a formal power series

\[
\phi(q) = \sum_{m \geq 0} d(m) q^m \in \mathbb{C}[[q]],
\]

the following are equivalent:

1. \( \phi(q) \) is the \( q \)-expansion of a modular form\(^9\) in \( M_\infty^k(D, \chi) \).
2. the relation

\[
\sum_{m \geq 0} c(-m) d(m) = 0
\]

for every weakly holomorphic form\(^10\)

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f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M_{2-k}^!(D, \chi).
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\(^9\)\( M_\infty^k(D, \chi) = \) cuspidal outside of \( \infty \).

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To apply this in our situation, we need to produce relations in \( \widehat{CH}^1_Q(S^*) \) of the form

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\sum_{m \geq 0} c(-m) \widehat{Z}^{\text{tot}}(m) = 0,
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f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M^{1,\infty}_{2-n}(D, \chi).
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Those familiar with the theory of Borcherds forms will recognize that, as a starting point, we will want to associate to such a weakly homomorphic form \( f \), with \( c(-m) \in \mathbb{Z} \) for \( m \geq 0 \), a meromorphic section \( \psi(f) \) of \( \omega^k_{\text{an}} \) on \( \text{Sh}(G, D)(\mathbb{C}) \).

This is the unitary group analogue of the Borcherds lift for \( \text{SO}(n - 2, 2) \).

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Stephen Kudla (Toronto) 

Arithmetic theta series
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Indeed, we have a morphism of Shimura varieties

\[ j : \text{Sh}(G, D)(\mathbb{C}) \longrightarrow \text{Sh}(\tilde{G}, \tilde{D})(\mathbb{C}) \]

where \( \tilde{G} = \text{SO}(V) \), for the rational quadratic space \( V \) with quadratic form defined by \( Q(x) = \langle x, x \rangle \). Thus, \( \text{sig}(V) = (2n - 2, 2) \).

For a weakly holomorphic form \( f \in \mathcal{M}_{2, \infty}^!(D, \chi) \), the Borcherds lift defines a meromorphic modular form \( \tilde{\psi}(f) \) on \( \text{Sh}(\tilde{G}, \tilde{D})(\mathbb{C}) \).

Let

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Of course, such a Borcherds lift could be defined directly for the dual pair \((U(1, 1), U(n - 1, 1))\), but it is more efficient to take advantage of the extensively developed theory for the dual pair \((\text{SL}(2), \text{O}(2n - 2, 2))\), via the seesaw.
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Unitary Borcherds forms

Note that the inequivalent cusps of $\Gamma_0(D)$ are $\infty_r \sim \frac{r}{D}$ where $r \mid D$.

For a weakly holomorphic form

$$f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M_{2-n}^!(D, \chi),$$

let $c_r(0)$ be its normalized constant term at the cusp $\infty_r$.
We can assume that $c(-m)$ for $m > 0$ and $c_r(0)$ all lie in $\mathbb{Z}$.

**Theorem A.** (1) Suitably normalized, $\psi(f) = j^* \tilde{\psi}(f)$ is a rational section of the line bundle $\omega^k$ on $S^*$, where $k = \sum_{r \mid D} c_r(0)$.
(2) The divisor of this section on $S^*$ is given by

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$$- \frac{1}{2} \sum_{m > 0} c(-m) \sum_{s \in \text{Sing}} | \{ x \in \text{Hom}_{O_k}(A_0, A_s) \mid \langle x, x \rangle = m \} | \cdot \text{Exc}_s.$$
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Thus, the divisor \( \text{div} \psi(f) \) involves

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(b) boundary divisors \( S^*(\Phi) \) at the various cusps \( \Phi \), with multiplicities

\[
\sum_{m > 0} c(-m) \frac{m}{n-2} \left| \{ x \in L_{\Phi} \mid \langle x, x \rangle = m \} \right|
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(c) components \( \text{Exc}_s \) of the exceptional locus for the blowup \( S^* \Rightarrow S^*_\text{Kra} \to S^*_\text{Pap} \supset \text{Sing} \), with multiplicities

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\left| \{ x \in L_s \mid \langle x, x \rangle = m \} \right|, \quad L_s = \text{Hom}_{O_k}(A_{0,s}, A_s)
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\[
\sum_{m>0} c(-m) \frac{m}{n-2} \left| \{ x \in L_\Phi \mid \langle x, x \rangle = m \} \right|
\]

(c) components \( \text{Exc}_S \) of the exceptional locus for the blowup
\( S^* = S_{\text{Kra}}^* \longrightarrow S_{\text{Pap}}^* \supset \text{Sing} \), with multiplicities

\[
\left| \{ x \in L_s \mid \langle x, x \rangle = m \} \right|, \quad L_s = \text{Hom}_{O_k}(A_{0,s}, A_s)
\]

(d) and multiples of the fibers \( S_p^* \) at ramified primes \( p \mid D \).
Thus, the divisor $\text{div} \, \psi(f)$ involves

(a) the Zariski closure of $\mathcal{Z}^*(m)$ of $\mathcal{Z}(m)$ in $S^*$

(b) boundary divisors $S^*(\Phi)$ at the various cusps $\Phi$, with multiplicities

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(c) components $\text{Exc}_s$ of the exceptional locus for the blowup $S^* = S^*_\text{Kra} \longrightarrow S^*_\text{Pap} \supset \text{Sing}$, with multiplicities

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(d) and multiples of the fibers $S_p^*$ at ramified primes $p \mid D$. 

---

Stephen Kudla (Toronto)

Arithmetic theta series
Green functions

The class of $\text{div } \psi(f)$ and of $\omega^k$ coincide in the Chow group $\text{CH}^1_{\mathbb{Q}}(S^*)$, but we still want to include the Green functions.

Following an idea due to Bruinier, consider the space of harmonic Maass forms:

$$H^\infty_{2-n}(D, \chi) \supset M^1_{2-n}(D, \chi).$$

These have expansions

$$f(\tau) = \sum_{m \gg -\infty} c^+(m) q^m + \sum_{m < 0} c^-(m) \Gamma(n - 1, 4\pi |m|v) q^m,$$

where $\tau = u + iv$ and $\Gamma(s, x) = \int_x^\infty e^{-t} t^{s-1} dt$.

For $m \in \mathbb{Z}_{>0}$, there is a unique such function $f_m$ with

$$f(\tau) = q^{-m} + O(1), \quad \text{as } q \to 0.$$
The class of $\text{div} \, \psi(f)$ and of $\omega^k$ coincide in the Chow group $\text{CH}^1_{\mathbb{Q}}(S^*)$, but we still want to include the Green functions.

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The class of $\text{div} \, \psi(f)$ and of $\omega^k$ coincide in the Chow group $\text{CH}_1^1(S^*)$, but we still want to include the Green functions.

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$$H_{2-n}^\infty(D, \chi) \supset M_{2-n}^{1,\infty}(D, \chi).$$

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One can take such forms as inputs in Borcherds regularized theta integral $\Theta^{\text{reg}}$. The crucial facts are:

1. $\Theta^{\text{reg}}(f_m)$ is a logarithmic Green function\(^{12}\) on $S^*(\mathbb{C})$ for the divisor $Z^{\text{tot}}(m)(\mathbb{C})$.

Therefore we can define

$$\hat{Z}^{\text{tot}}(m) = (Z^{\text{tot}}(m), \Theta^{\text{reg}}(f_m)) \in \widehat{\text{CH}}_{\mathbb{Q}}^1(S^*)$$

2. If $f \in M^{l, \infty}_{2-n}(D, \chi)$ is weakly holomorphic, then

$$\Theta^{\text{reg}}(f) \equiv -\log \|\psi(f)\|^2.$$ (up to log – log-negligible terms)

where $\| \cdot \|$ is the norm on $\hat{\omega}$.

\(^{12}\)Bruinier-Howard-Yang

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With this definition of the classes $\hat{Z}^{\text{tot}}(m)$ and using (2), we have the relation

$$\hat{\omega}^k \equiv \hat{\text{div}} \psi(f) := (\text{div} \psi(f), -\log \|\psi(f)\|^2), \quad \text{in } \hat{\text{CH}}_1(Q(S^*)),$$

where $k = k(f) = \sum_{r|D} c_r(0)$ depends on $f$.

Now we do some bookkeeping and use the fact that

$$c_r(0) = -\sum_{m>0} c(-m) e_r(m),$$

where the Eisenstein series associated to the cusp $\infty_r$ has Fourier expansion

$$E_r(\tau) = \sum_{m \geq 0} e_r(m) q^m.$$
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where the Eisenstein series associated to the cusp $\infty_r$ has Fourier expansion

$$E_r(\tau) = \sum_{m \geq 0} e_r(m) q^m.$$
By the modularity criterion, it follows that the series

\[
\hat{\phi}(\tau) - \frac{1}{2} \sum_{s \in \text{Sing}} \theta(\tau; L_s) \cdot \text{Exc}_s \\
+ (\hat{\omega} - \frac{1}{2} \text{Exc}) \cdot \sum_{r \mid D} E_r(\tau) + \sum_{p \mid D} S_p^* \cdot \sum_{r \mid D \atop p \nmid r} E_r(\tau)
\]

is a modular form of weight \( n \), character \( \chi \), and level \( D \), valued in \( \hat{\text{CH}}_1^1(S^*) \).

Hence so is \( \hat{\phi}(\tau) \), as claimed.
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Hence so is $\hat{\phi}(\tau)$, as claimed.
Computation of the divisor

It remains to explain something about the proof of Theorem A, in particular, about the determination of $\text{div}\psi(f)$ on the integral model $S^*$.

As there are many technical issues, let me just describe the main strategy: We study $\psi(f)$ and $\tilde{\psi}(f)$ in a neighborhood of the boundary. First consider the complex situation:

$$\text{Sh}(G, D)(\mathbb{C}) \xrightarrow{j} \text{Sh}(\tilde{G}, \tilde{D})(\mathbb{C})$$

$J$ = isotropic $k$-line in $V$ $\implies$ isotropic $\mathbb{Q}$-plane $J$ in $V$.

point boundary component $\implies$ curve boundary component

so that, for the Baily-Borel compactifications

$$\text{Sh}(G, D)(\mathbb{C})^{BB} \xrightarrow{j} \text{Sh}(\tilde{G}, \tilde{D})(\mathbb{C})^{BB}$$

we have a CM point mapping to a modular curve.
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Computation of the divisor

In the smooth toroidal compactifications, these are blown up to

\[
\left( \mathcal{L}_\Phi^{-1} \longrightarrow B_\Phi \longrightarrow M_{1,0} \right) \quad \text{CM-point} \quad \left( \tilde{\mathcal{L}}_\Phi^{-1} \longrightarrow \mathcal{KS}_\Phi \longrightarrow \mathcal{Y}_0(D) \right)
\]

where \( \mathcal{KS}_\Phi \longrightarrow \mathcal{Y}_0(D) \) is a Kuga-Sato variety over a modular curve.

Borcherds gave a product formula for \( \tilde{\psi}(f) \) valid in a neighborhood of a point boundary component on \( \tilde{D} \).

There is another product formula for \( \tilde{\psi}(f) \), valid in a neighborhood of a curve boundary component.

From this product formula, we can read off the Fourier-Jacobi expansion of \( \psi(f) \) on the formal completion of \( (\mathcal{L}_\Phi^{-1})_{B_\Phi} \),

\[
\psi(f) = q^{\text{mult}_\Phi(f)} \psi_0 \sum_{\ell \geq 0} \psi_\ell \cdot q^{\ell}, \quad \psi_0 = \text{leading FJ coeff.}
\]

\[
\text{mult}_\Phi(f) = \sum_{m > 0} \frac{c(-m)}{n - 2} \left| \left\{ x \in L_\Phi \mid \langle x, x \rangle = m \right\} \right|
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In the smooth toroidal compactifications, these are blown up to

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where \( \mathcal{K}S_\Phi \rightarrow \mathcal{Y}_0(D) \) is a Kuga-Sato variety over a modular curve. Borcherds gave a product formula for \( \tilde{\psi}(f) \) valid in a neighborhood of a point boundary component on \( \tilde{\mathcal{D}} \).

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Computation of the divisor

The divisor of $\psi(f)$ on $(L_\Phi^{-1})_{B_\Phi}$ is then the pullback $\pi^*(\text{div}(\psi_0))$, 

$$
\begin{array}{ccc}
L_\Phi^{-1} & \xrightarrow{\pi} & B_\Phi \\
\uparrow & & \uparrow \\
\text{div}(\psi) & & \text{div}(\psi_0)
\end{array}
$$

where $\psi_0$ is the leading Fourier-Jacobi coefficient.

Note that $\psi_0$ is a rational section of a certain line bundle $\omega_\Phi^k \cdot L_\Phi^{\text{mult}_\Phi(f)}$ on $B_\Phi = L_\Phi \otimes E$.

Also, any nonzero vector $x \in L_\Phi$ defines a homomorphism $j_x : B_\Phi = L_\Phi \otimes E \rightarrow E, \langle x, \cdot \rangle$. 
The divisor of $\psi(f)$ on $(\mathcal{L}_\Phi^{-1})^{\hat{A}}_{B_{\Phi}}$ is then the pullback $\pi^*(\text{div}(\psi_0))$,

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Also, any nonzero vector $x \in \mathcal{L}_\Phi$ defines a homomorphism

$$j_x : \mathcal{B}_\Phi = \mathcal{L}_\Phi \otimes E \longrightarrow E, \quad \langle x, \cdot \rangle.$$
Finally, the product formula for $\psi(f)$ shows that

$$\psi_0 = P^\eta_\Phi \cdot P^{\text{vert}}_\Phi \cdot P^{\text{hor}}_\Phi$$

where $P^\eta_\Phi$ is a CM-value of a power of the Dedekind $\eta$-function, and

$$P^{\text{vert}}_\Phi = \prod_{r \mid D} \prod_{b \in \mathbb{Z}/D\mathbb{Z}} \Theta(\tau, \frac{b}{D})^{c_r(0)}, \quad \tau = \text{CM-point}$$

$$P^{\text{hor}}_\Phi = \prod_{m > 0} \prod_{\substack{x \in L_\Phi \quad Q(x) = m}} \Theta(\tau, \langle w_0, x \rangle)^{c(-m)},$$

where

$$\Theta(\tau, z) = \frac{i^\frac{\vartheta_1(\tau, z)}{\eta(\tau)}}{\eta(\tau)} = q^{\frac{1}{12}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1 - \zeta q^n)(1 - \zeta^{-1} q^n).$$

These are the formulas over $\mathbb{C}$, but the Jacobi theta function lives over $\mathbb{Z}$ and eventually we arrive at Theorem A.
Computation of the divisor

Finally, the product formula for $\psi(f)$ shows that

$$\psi_0 = P_\phi^\eta \cdot P_\phi^{\text{vert}} \cdot P_\phi^{\text{hor}}$$

where $P_\phi^\eta$ is a CM-value of a power of the Dedekind $\eta$-function, and

$$P_\phi^{\text{vert}} = \prod_{r|D} \prod_{b \in \mathbb{Z}/D\mathbb{Z} \atop b \neq 0} \Theta(\tau, \frac{b}{D})^{c_r(0)}, \quad \tau = \text{CM-point}$$

$$P_\phi^{\text{hor}} = \prod_{m > 0} \prod_{x \in L_\phi \atop Q(x) = m} \Theta(\tau, \langle w_0, x \rangle)^{c(-m)}, \quad \text{where}$$

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These are the formulas over $\mathbb{C}$, but the Jacobi theta function lives over $\mathbb{Z}$ and eventually we arrive at Theorem A.