Joint Equidistribution of CM Points

Ilya Khayutin

September 29, 2017

André-Oort for Product of Modular Curves

Theorem (Pila '11 for general n, André '98 for $n = 2$. Conditionally on GRH: Edixhoven '98 and Edixhoven '05)
Let $X = \underbrace{Y \times \ldots \times Y}_{n}$ *be the Cartesian power of a modular curve. Assume* $\{x^i = (x^i_1, \ldots, x^i_n)\}_i$ *is a sequence of special points in* X*, i.e. each* $x^i_k \in Y_k$ *is a CM point. If the intersection of this sequence with any proper special subvariety is finite then this sequence is Zariski dense in* X*.*

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- A special point (x_1, x_2),
- $\{x\} \times Y$ and $Y \times \{x\}$ for $x \in Y$ a CM point,
- image of a Hecke correspondence $T_n \hookrightarrow Y \times Y$, e.g. the diagonal embedding $Y \overset{\Delta}{\hookrightarrow} Y \times Y$.

Equidistribution Conjecture

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Let $\{x_i\}_i$ be a generic sequence of special points in X – the Cartesian power of a modular curve. Let the probability measure μ_i on X be the normalized counting measure on the Galois orbit of x_i

$$\mu_i := \frac{1}{|\mathrm{Orb}(x_i)|} \sum_{y \in \mathrm{Orb}(x_i)} \delta_y$$

Then $\{\mu_i\}_i$ converges weak-$*$ to the uniform measure $m_X = \underbrace{m_Y \times \ldots m_Y}_{n}$.

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Asymptotic density of Galois orbits in the locally compact topology.

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- $n = 1$ **–Duke ’88, Iwaniec ’87... Michel ’04, SW Zhang ’05**. Assuming a split prime: **Linnik ’55**.

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Then $\mu_i^{\mathrm{joint}} \to m_Y \times m_Y$.

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