Boundary strata and adjoint varieties[1]

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IAS, October 15, 2014

[1] report on recent work with C. Robles, based in part on earlier work with G. Pearlstein as well as P. Griffiths and M. Green.

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smooth reps. of classes in $H^*(\check{D},\mathbb{Z})$ $\leftarrow - -$ enhanced SL_2-orbits

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We think of M-T domains as parametrizing (a connected component of) all HS on V polarized by Q, with the same Hodge numbers as φ, whose HTs include the fixed tensors of G. We shall loosely speak of (V, Q, φ) as a "Hodge representation" of G.

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- Since $-B$ is > 0 on $\mathfrak{k}$ and < 0 on $\mathfrak{k}^{\perp}$, $(\mathfrak{g}, \mathrm{Ad} \circ \varphi, -B)$ is a PHS (of weight 0).

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Since φ is "sufficiently general", we may take $g_0 = 1$ in (c).

By (a), $M \trianglelefteq G$; so G simple $\implies M = G$.

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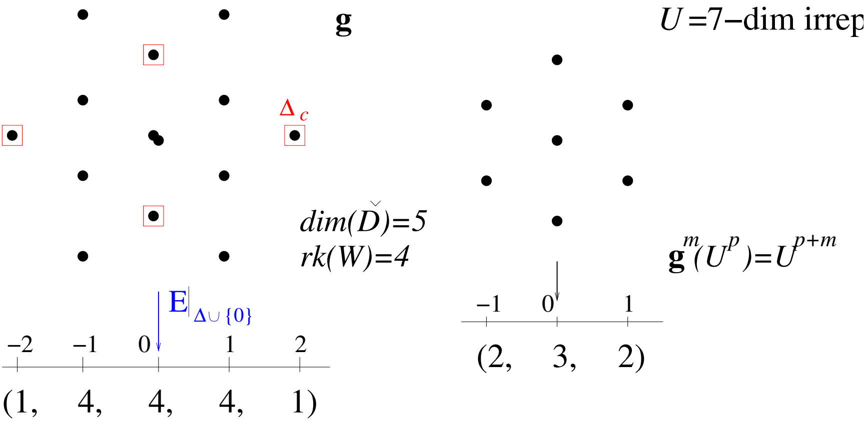
Example: $(G = G_2)$



Generalization to fundamental adjoint varieties

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- the $G(\mathbb{C})$-orbit of $[U_\mu] \in \mathbb{P}U$ gives a homogeneous embedding of $G(\mathbb{C})/P$, minimal if $\mu^i \in \{0, 1\}$ $(\forall i)$.

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Why study the adjoint varieties as Hodge-theoretic classifying spaces?
One reason: they are the "simplest" G/P with nontrivial IPR, in the sense of being precisely the cases where $\mathcal{W}$ is a contact distribution.

§2. Schubert VHS and classical subdomains

Let $D \subset \check{D}$ be a M-T domain with base point $F^\bullet = F^\bullet_\varphi \in D$, $\mathfrak{g} = \oplus \mathfrak{g}^j$ the corresponding Hodge decomposition, and T, Δ as before. Write $P \geq B \geq T_{\mathbb{C}}$ for the parabolic fixing $F^\bullet$, so that we have:

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- X_w Schubert VHS (i.e. horizontal) $\iff \mathfrak{n}_w \subset \mathfrak{g}^{-1}$
  $(\implies \mathfrak{n}_w$ abelian$)$

Theorem (Robles)

$$\max\dim(\mathrm{IVHS}) = \max\dim(\mathrm{SVHS}) = \max\left\{|\Delta_w| \,\middle|\, w \in W^P,\ \Delta_w \subset \Delta(\mathfrak{g}^{-1})\right\}$$

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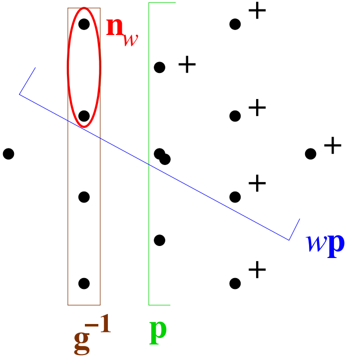
Example : ($G = G_2$)

There exists only one X_w of dimension 2, and it is an SVHS (false for higher dim.)

$\Downarrow$

the maximal integral manifold of $\mathcal{W}$ has dimension 2



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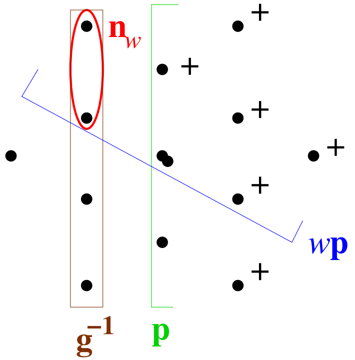

Is this X_w a M-T subdomain? If so, it would be a (smooth, Hermitian) homogeneous $G'(\mathbb{R})$-orbit, with $\mathfrak{g} \supsetneq \mathfrak{g}' =$ Lie algebra closure of $\mathfrak{n}_w \oplus \overline{\mathfrak{n}_w}$. But this closure is all of $\mathfrak{g}$. NO!

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It is instructive to compare this with another recent result:

Theorem (Friedman-Laza)

- $X \subset \check{D}$ a smooth "VHS" (horizontal subvariety)
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The converse of the Corollary is false: there are plenty of non-Schubert, horizontal Hermitian M-T subdomains, and we will construct maximal integral ones later.

§3. Lines on $\check{D}$ and a differential invariant of VHS

The Corollary suggests that there might be lots of singular SVHS, in view of the G_2 example. A systematic construction of Schubert varieties is given by incidence correspondences:

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G(\mathbb{C})/Q & & G(\mathbb{C})/P \arrow[r, equal] & \check{D} \\
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- $\mathsf{X}(Q/Q) = X(\mathscr{D}')$, where $\mathscr{D}' = \mathscr{D} \setminus (I(\mathfrak{q}) \setminus I(\mathfrak{p}))$

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- $X_0 \cong \mathrm{Cone}(\mathcal{C}_0)$ is a singular Schubert VHS

Some data for the fundamental adjoint varieties $\check{D}$ and their associated "subadjoint" varieties $\mathcal{C}_0$ of lines through a point:

$\mathfrak{g}_{\mathbb{C}}$	$\check{D}$	$\mathfrak{g}_{\mathbb{C}}^{0,ss}$	$\mathcal{C}_0$
$\mathfrak{so}(n)$	$OG(2,\mathbb{C}^n)$	$\mathfrak{so}(n-4)\oplus\mathfrak{sl}(2)$	$\mathbb{P}^1\times\mathcal{Q}^{n-6}$
$\mathfrak{e}_6$	E_6/P_2	$\mathfrak{sl}(6)$	$Gr(3,\mathbb{C}^6)$
$\mathfrak{e}_7$	E_7/P_1	$\mathfrak{so}(12)$	$\mathcal{S}_6$
$\mathfrak{e}_8$	E_8/P_8	$\mathfrak{e}_7$	E_7/P_7
$\mathfrak{f}_4$	F_4/P_1	$\mathfrak{sp}(6)$	$LG(3,\mathbb{C}^6)$
$\mathfrak{g}_2$	G_2/P_2	$\mathfrak{sl}(2)$	$\nu_3(\mathbb{P}^1)$

We now relate these varieties of lines to the Griffiths-Yukawa kernel. Let $\mathcal{V} = \oplus_{j=0}^{n} \mathcal{V}^{n-j,j}$ be a VHS over $\mathcal{S}$, with associated period map $\Phi : \mathcal{S} \to \Gamma \backslash D$ ($D =$ M-T domain). Denote by $\mathcal{D}$ a holomorphic differential operator on $U \subset \mathcal{S}$ of order n.

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The composition

$$\mathcal{O}_U(\mathcal{V}^{n,0}) \hookrightarrow \mathcal{O}_U(\mathcal{V}) \xrightarrow{\mathcal{D}} \mathcal{O}_U(\mathcal{V}) \twoheadrightarrow \mathcal{O}_U(\mathcal{V}^{0,n})$$

depends only on $\sigma(\mathcal{D})$, giving rise to the G-Y coupling

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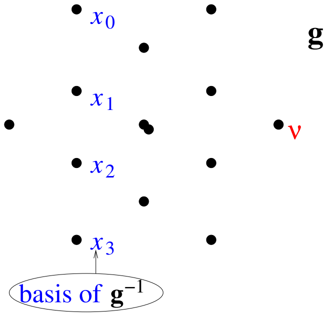
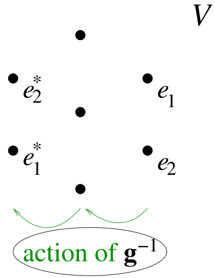
Write $\mathcal{Y} \subset \mathbb{P}\mathfrak{g}^{-1}$ for the kernel of $(*)$ (at $F^\bullet \in D$).

Example : $(G = G_2)$ $V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ 7-diml irrep





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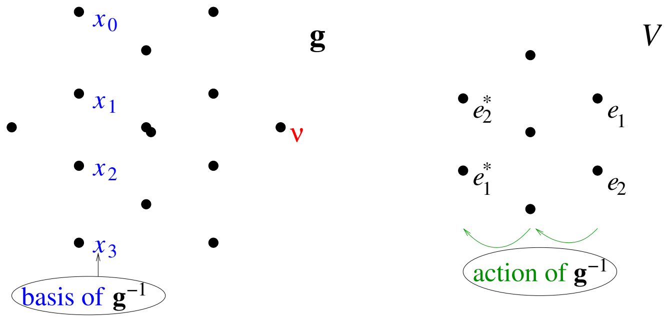

Given $\xi := \sum \xi_i x_i \in \mathfrak{g}^{-1}$, one computes

$$ {}_{e^*}[\xi^2]_e = \begin{pmatrix} -2\xi_1\xi_2 + 2\xi_2^2 & \xi_1\xi_2 - \xi_0\xi_3 \\ \xi_1\xi_2 - \xi_0\xi_3 & -2\xi_0\xi_2 + 2\xi_1^2 \end{pmatrix} $$

whose vanishing defines the twisted cubic $\nu_3(\mathbb{P}^1) \subset \mathbb{P}\mathfrak{g}^{-1}$.

This example is generalized to E_6, E_7, F_4 by the

Theorem (K-R)

For $\check{D} = G(\mathbb{C})/P$ a fundamental adjoint variety, and V a level 2 Hodge representation such that $V^{2,0}$ is a faithful representation of $\mathfrak{g}^0$, we have $\mathcal{Y} = \mathcal{C}_0$. (Much more generally, $\mathcal{Y}$ contains the horizontal lines through $F^\bullet$.)

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Sketch: Given $[\xi] \in \mathcal{Y}$, $\xi^2(u) = 0 \; \forall u \in V^{2,0}$.

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Sketch: Given $[\xi] \in \mathcal{Y}$, $\xi^2(u) = 0 \ \forall u \in V^{2,0}$. Fix $v \in \mathfrak{g}^2 \setminus \{0\}$, so $\mathrm{ad}_\xi^2 v \in \mathfrak{g}^0$.

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When the conclusion of the Theorem holds,

- $\mathcal{Y} = \ker(\text{G-Y})$ gives Hodge-theoretic meaning to $\mathcal{C}_0$
- $\mathcal{C}_0 \cong G^0(\mathbb{C})/\cdots$ gives a homogeneous description of $\mathcal{Y}$
- $\mathrm{ad}_\xi^2 v = 0$ produces explicit projective homogeneous equations for both.

§4. $G(\mathbb{R})$-orbits in $\check{D}$ and asymptotics of VHS

Given the input:

- $D \subset \check{D}$ M-T domain (parametrizing wt. 0 HS on V)
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we define (and assume nonempty):

- $\tilde{B}(\sigma) := \left\{ F^\bullet \in \check{D} \;\middle|\; \begin{array}{l} e^{\sum t_i N_i} F^\bullet \in D \quad \text{for } Im(\tau_i) \gg 0 \\ N_i F^\bullet \subset F^{\bullet - 1} \end{array} \right\}$

which parametrizes LMHS $(F^\bullet, W(\sigma)_\bullet)$,[2]

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One may "partially compactify" $\Gamma \backslash D$ by $\bar{B}(\sigma)$s (log manifold).

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Structure of $B(\sigma)$

Write

- $M_\sigma = \exp\{\mathrm{im}(\sum N_i) \cap (\cap \ker(N_i))\}$
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- naive limit map

$$\begin{array}{cccc} \Phi_\infty^\sigma : & B(\sigma) & \to & \partial D \subset \check{D} \\ & F^\bullet & \mapsto & \lim\limits_{Im(\tau)\to\infty} e^{\tau N} F^\bullet \quad (\text{any } N \in \sigma^\circ). \end{array}$$

[K-R] contains a general prescription for using a set $\mathfrak{B} = \{\beta_1, \ldots, \beta_s\} \subset \Delta(\mathfrak{g}^1)$ of strongly orthogonal roots to explicitly construct $\mathbb{Q}$-split $(\sigma, F_0^\bullet)$. The motivation is to parametrize $G(\mathbb{R})$-orbits in ∂D in the image of Φ_∞^σ. I will discuss only $s = 1$. Fix a base point $o \,(\longleftrightarrow F^\bullet)$ in D.

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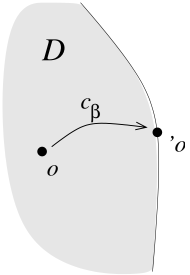
- $\mathtt{E} \rightsquigarrow {}'\mathtt{E}$
- $o \rightsquigarrow {}'o$

$(F^\bullet \rightsquigarrow {}'F^\bullet)$



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- If $\check{D} = G(\mathbb{C})/P$ (P maximal) and $\dim X(N) \geq 2$, then $X(N)$ is not Schubert.

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There is a unique codimension-1 $G(\mathbb{R})$-orbit in ∂D.

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- [K-P] $\mathbb{R}$-codim. of orbit $\ni {}'o$ is given by $\sum_{p,q>0} h^{p,q}$;
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§5. "Minimal" boundary of adjoint varieties

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$\implies q(\alpha_i) \geq 1 \implies$ codim > 1 unless $\beta = \alpha_i$. $\square$

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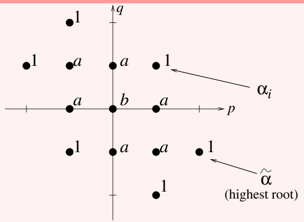

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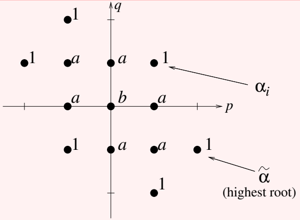

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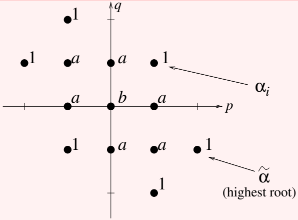

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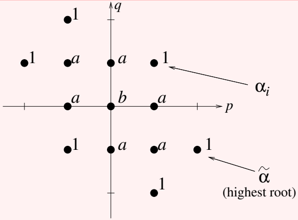

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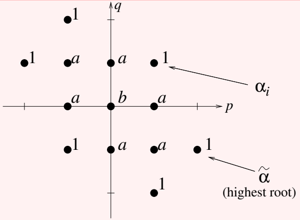

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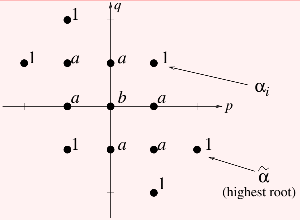

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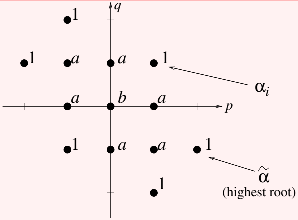

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$\implies w(\mathfrak{g}^{p,q}) = w(\mathfrak{g}^p \cap \mathfrak{g}_{p+q}) = \mathfrak{g}_{-p} \cap \mathfrak{g}^{-(p+q)} = \mathfrak{g}^{-p-q,q}$. $\square$

Note that w identifies the (faithful) representations of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ and $\mathfrak{g}^0$ on $\mathfrak{g}^{-1}$. Moreover, $D(N)\,(\subset \mathbb{P}\mathfrak{g}_1)$ is the M-T domain for the Hodge representation of G_0 on $\mathfrak{g}_1$, which leads to:

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Applications? Automorphic cohomology; geometric realizations; cohomology of $\check{D}$.

$H^*(\check{D}, \mathbb{Z})$ is generated by Schubert vareities, and the "horizontal" part (invariant characteristic cohomology) by Schubert VHS. Do the subadjoint cylinder classes $[X(N)]$ yield smooth representatives of the subadjoint cone classes $[X_w]$?

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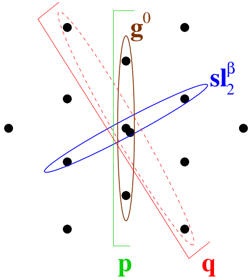

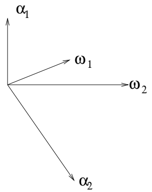

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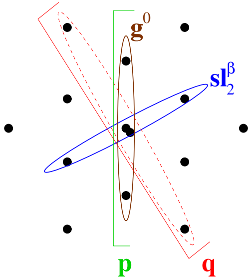

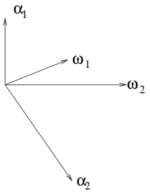

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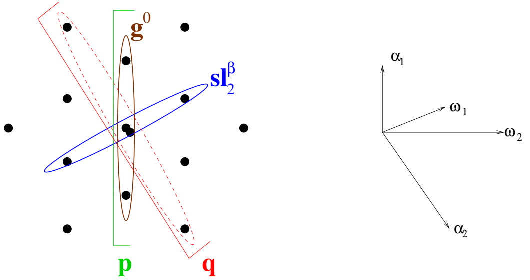

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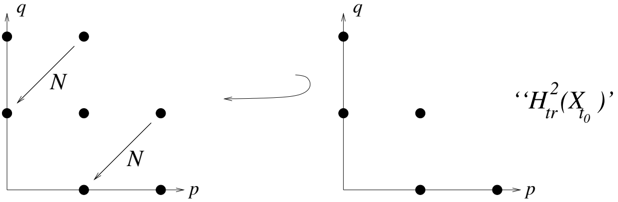

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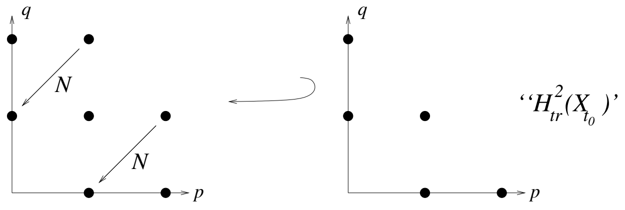

where N is the monodromy logarithm and bullets denote 1-dimensional spaces. In fact, just such a family has been constructed by N. Katz using elliptic fibrations; the M-T group is determined by a moment computation using elliptic convolution over finite fields. We shall describe a special case.

Begin with the rational elliptic surface

$$\mathcal{E} \to \mathbb{P}^1_z : \; y^2 = x(1-x)(x-z^2)$$

with singular fibers $(2\, I_4,\, 2\, I_2)$ at $z = -1, 0, 1, \infty$.

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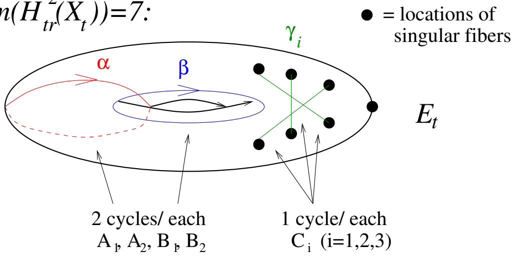
$dim(H^2_{tr}(X_t))=7$:



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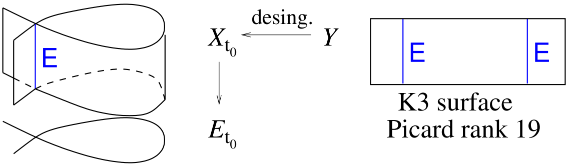

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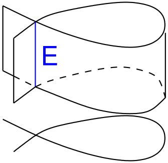

$$\begin{tikzcd} X_{t_0} \arrow[d] & Y \arrow[l, "\text{desing.}"'] \\ E_{t_0} & \end{tikzcd}$$

K3 surface
Picard rank 19

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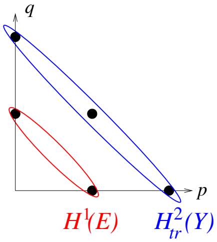
The part of $H^2(X_{t_0})$ not coming from the 19 algebraic classes on Y indeed takes the form



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- The point in the fiber

$$\begin{aligned}\mathbb{C}^2/\mathbb{Z}\langle \tbinom{1}{0}, \tbinom{2\tau/3}{1}, \tbinom{\tau^2/3}{2\tau}, \tbinom{0}{3\tau^2} \rangle &\cong J(Sym^3 H^1(\mathrm{E})) \\ &\subset J(H^2_{tr}(Y)^\vee \otimes H^1(E))\end{aligned}$$

  is given by $\int_{B_1} \omega,\ \int_{B_2} \omega$ $(\omega \in \Omega^2(Y))$.

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To determine which deformations of X_t "preserve G_2", it may be necessary to "see" the cubic Hodge tensor geometrically: we need $\mathfrak{Z} \in CH^3(X_t \times X_t \times X_t)$ inducing an "octonionic cross-product" on $H^2_{tr}(X_t)$.

– Thank You –