Uniqueness aspects of symplectic fillings

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Definition

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**Example**

1. $\mathbb{R}^{2n+1}$ with $\xi := \ker(\alpha = dz - \sum_{i=1}^{n} p_i dq_i)$.
2. $S^{2n-1}$ with $\xi_{std} := JTS^{2n-1} \cap TS^{2n-1}$.
3. $ST^*M$, links of singularities, etc.
Figure: Standard contact $\mathbb{R}^3$
Symplectic fillings

Definition

\((W, \lambda)\) is called a Liouville filling of contact manifold \((Y, \xi)\) iff \(\partial W = Y\) and the following holds

1. \(\omega := d\lambda\) is a symplectic form on \(W\) and there is a vector field \(X\) such that \(L_X\omega = \omega\) and \(X\) is pointing out along boundary.

2. \(\xi = \ker \lambda\).

Example

1. The standard symplectic ball is a filling of the standard contact sphere.

2. The cotangent disk bundle \(DT^*M\) is a filling of \(ST^*M\).

3. Smoothings of a singularity are fillings of the link of the singularity.
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Symplectic cobordisms

In general, we have Liouville cobordisms between contact manifolds.

Question

*Understand this cobordism category.*
Symplectic field theory (SFT) is a field theory on the cobordism category of contact manifolds, a special case of SFT assigns an algebra $CC_*(Y)$ called contact homology to a contact manifold $Y$. For the empty contact manifold, the assigned algebra is the ground field $k$. Therefore given a filling $W$ of $Y$, we have a map $\epsilon_W : CC_*(Y) \to k$, such map is called an augmentation.

It is much easier to classify augmentations than to classify fillings.

Slogan
If a contact manifold admits only the trivial augmentation, then many Floer theoretic properties and topological properties of the filling is independent of the filling.
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A contact manifold is called asymptotically dynamically convex (ADC) if the SFT gradings are positive.

Theorem (Lazarev)
The ADC property is preserved under subcritical and flexible handle attachment.
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**Example**
1. (Lazarev) Boundary of flexible Weinstein domain $W$ with $c_1(W) = 0$.
2. $ST^*M$ when $\text{dim } M \geq 4$.
3. (McLean) Links of terminal singularities.
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The same method was used by Oancea-Viterbo and Barth-Geiges-Zehmisch to reach the following.

Theorem (Barth-Geiges-Zehmisch)

Let \(Y\) be a simply connected subcritically fillable contact manifold of dimension \(\geq 5\), then exact fillings of \(Y\) have the unique diffeomorphism type.
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The strategy of proof is finding a "homological foliation" by pseudo-holomorphic curves, the existence of foliation is hinted by the fact that \(W = V \times \mathbb{C}\), when \(W\) is subcritical.
Uniqueness of symplectic fillings

For every Liouville domain $W$, we can assign it with two Floer type theories $SH^*(W)$, $SH^+_+(W)$, so that they fit into a long exact sequence,

$$\ldots \rightarrow H^*(W) \rightarrow SH^*(W) \rightarrow SH^+_+(W) \rightarrow H^{*+1}(W) \rightarrow \ldots.$$
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**Theorem (Z 19')**

Let $Y$ be ADC, then $\delta_\partial : SH_+^*(W) \rightarrow H^{*+1}(W) \rightarrow H^{*+1}(Y)$ is independent of the filling $W$ as long as $c_1(W) = 0$ and $\pi_1(Y) \rightarrow \pi_1(W)$ is injective.
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A quick corollary is that if $Y$ is flexibly fillable, then $H^*(W) \rightarrow H^*(Y)$ is independent of filling.
Since $1 \in \text{Im} \, \delta$ is equivalent to $SH_\ast(W) = 0$, therefore for ADC contact manifolds, the vanishing of symplectic cohomology is independent of filling.
Generalizations

- Since $1 \in \text{Im} \delta_{\partial}$ is equivalent to $SH_*(W) = 0$, therefore for ADC contact manifolds, the vanishing of symplectic cohomology is independent of filling.

- The vanishing of symplectic cohomology is the first symplectic property measuring the complexity of symplectic domains in a whole hierarchy. The next one is the existence of dilation, i.e. $\exists x \in SH^1(W)$ such that $\Delta(x) = 1$. In general, we can consider the $S^1$-equivariant symplectic cohomology, we say $W$ carries a $k$-dilation, if $1$ is killed in the $k + 1$-th page of the spectral sequence from the $u$-filtration.
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Theorem (Z 19’)

There are structure maps related to the existence of $k$-dilation. When $Y$ is ADC, and all of them are independent of the filling $W$ as long as $c_1(W) = 0$ and $\pi_1(Y) \to \pi_1(W)$ is injective.
An instant corollary of the previous theorem is that the existence of $k$-dilation is independent of filling, if the boundary is ADC.
Applications

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- If $Y$ is ADC, then whether $\text{Im} \, \delta_{\partial}$ contains an element of degree $> \frac{\dim Y + 1}{2}$ is an obstruction to Weinstein fillability. The obstruction is symplectic in natural, in particular, there are infinitely many $4k + 3$ contact manifolds that are exactly fillable, almost Weinstein fillable, but not Weinstein fillable.
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- The $k$-dilation can be used to define a cobordism obstruction for ADC manifolds.
Further questions

1. Reformulising the constructions using SFT, so that the slogan can be made rigorous once the construction for SFT is completed.

2. The existence of \( k \)-dilation implies uniruledness, it is unlikely that uniruledness will imply the existence of \( k \)-dilation for some \( k \) for affine varieties. Is there a symplectic characterization of uniruledness for affine varieties?

3. The \( k \)-dilation gives a rough classification of symplectic domains with log Kodaira dimension \( -\infty \), what about other log Kodaira dimensions?
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without filling

overtwisted

Giroux torsion

k-algebraic torsion
starts to have fillings
but maybe unique filling
flexibly fillable
vanishing of symplectic cohomology
symplectic dilation
k-dilation
log-Kadaira negative
quasi dilation

quasi cyclic dilation

Log Calabi-Yau

positive log Kodaira
log general

with infinite fillings
empty set
Thank you!