Diffusion in high Sobolev spaces for Hamiltonian PDEs

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Nonlinear Schrödinger Equation

- Nonlinear Schrödinger Equation:

\[
\begin{align*}
    i\partial_t u + \Delta u &= |u|^2 u, \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \quad u(t, x) \in \mathbb{C} \\
    u(0) &= u_0
\end{align*}
\]  

- Limit of the quantum dynamics of many-body systems, model in nonlinear optics, water waves

- Energy and mass conservation:

\[
\begin{align*}
    E(u(t)) : &= \frac{1}{2} \int |\nabla u(t, x)|^2 + \frac{1}{4}|u(t, x)|^4 dx = E(u(0)), \\
    M(u(t)) : &= \int |u(t, x)|^2 dx = M(u(0))
\end{align*}
\]

- If \( d = 1 \), NLS is completely integrable \( \implies \) all integer Sobolev norms stay bounded in time
Nonlinear Schrödinger equation on $\mathbb{T}^2$

- **Bourgain (1993):** If $u(0) \in H^s(\mathbb{T}^2)$ with $s \geq 1 \implies$ there exists a unique global-in-time solution such that $u(t) \in H^s$ for all $t$

- **Question:** what is the behavior of solutions as $t \to \infty$?

- **Bourgain (1996), Staffilani (1997):** $\|u(t)\|_{H^s} < Ct^{C(s-1)}$ as $t \to \infty$

- **Further question:** Is there any solution $u$ such that $\sup_t \|u(t)\|_{H^s} = \infty$? What would be the rate of growth?

- **Conjecture (Bourgain):** $\|u(t)\|_{H^s} \ll t^\varepsilon \|u(0)\|_{H^s}$ for all $\varepsilon > 0$
Forward energy cascade

- “forward energy cascade”: energy moves from lower frequencies to higher and higher frequencies

- growth of high Sobolev norms captures the energy cascade

\[
\lim_{t \to \infty} \| u(t) \|_{H^s} = \lim_{t \to \infty} \| \langle \xi \rangle^s \mathcal{F} u(t, \xi) \|_{L^2} = \infty \text{ for } s \text{ large}
\]

- in the physical space: dynamics moves to smaller and smaller scales causing a chaotic behaviour

- growth of high Sobolev norms is the minimal necessary condition for weak turbulence theory

- weak turbulence is the out-of-equilibrium statistics of random waves, it appeared in plasma physics, water waves (Zakharov (’60s))
Partial results

- **Bourgain** (1995, 1996): infinite time growth for examples of NLS and NLW (specific nonlinearity or specific perturbation of the Laplacian)

- **Kuksin** (1997): finite time growth for cubic NLS on $\mathbb{T}^d$, $d = 1, 2, 3$ with small dispersion

- **CKSTT** (2010): Cubic NLS on $\mathbb{T}^2$: For any $s > 1$, $\varepsilon \ll 1$, $K \gg 1$ there exists a solution $u(t)$ and $T > 0$ such that

  \[ \|u(0)\|_{H^s} \leq \varepsilon \text{ while } \|u(T)\|_{H^s} \geq K \]

- **Hani** (2011): infinite time growth for NLS on $\mathbb{T}^2$ with a truncated cubic nonlinearity

- **Hani, Pausader, Tzvetkov, Visciglia** (2013): infinite time growth for cubic NLS on $\mathbb{R} \times \mathbb{T}^d$, $d = 2, 3, 4$

- **Guardia, Kaloshin** (2012): $\|u(t)\|_{H^s} \geq K\|u_0\|_{H^s}$ for $0 \leq T \leq K^c$
Cubic half wave equation

- Cubic half wave equation:

\[
\begin{aligned}
&i\partial_t v - |D| v = |v|^2 v, \quad x \in \mathbb{R}, \quad v(t, x) \in \mathbb{C} \\
v(0) = v_0
\end{aligned}
\]

(NLW)

where \( F(|D|v)(\xi) = |\xi|Fv(\xi) \)

- Majda, McLaughlin, Tabak (1997): one dimensional models of weak turbulence:

\[
i\partial_t v - |D|^{\alpha} v = |D|^{-\beta/4} \left( |D|^{-\beta/4} v \right)^2 |D|^{-\beta/4} v, \quad 0 < \alpha < 1
\]

- For \( v_0 \in H^s(\mathbb{R}) \), \( s \geq \frac{1}{2} \), NLW has unique global-in-time solution such that \( v(t) \in H^s \) for all \( t \)

- Pocovnicu (2011): CKSTT-type of result: For any \( s > \frac{1}{2} \), \( \delta \ll 1 \) there exists a solution \( v(t) \) of NLW on \( \mathbb{R} \) such that

\[
\|v(0)\|_{H^s} \leq \delta, \text{ while } \|v(T)\|_{H^s} \geq \frac{1}{\delta} \text{ for } T = \left( \frac{1}{\delta} \right)^{\frac{s}{\alpha}} e^{\frac{2s-1}{s} \left( \frac{1}{\delta} \right)^{\frac{2}{\alpha}}}
\]
Resonant dynamics of NLW

- Birkhoff normal form / Renormalization group method yield the resonant dynamics

\[
|\xi| - |\xi_1| + |\xi_2| - |\xi_3| = 0
\]
\[
\xi - \xi_1 + \xi_2 - \xi_3 = 0
\]

\[\Rightarrow \xi, \xi_1, \xi_2, \xi_3 \text{ have the same sign}\]

- Resonant dynamics - Szegő equation:

\[i \partial_t u = \Pi_+ (|u|^2 u), \text{ where } \mathcal{F}(\Pi_f f)(\xi) = 1_{\xi \geq 0} \mathcal{F}f(\xi)\]

- Szegő equation was introduced by P. Gérard and S. Grellier in 2008

- Approximation result: NLW and Szegő equation with the same initial condition \(v_0 = u_0 \in H^s_+, \text{ of order } \varepsilon. \text{ Then:} \)

\[\|v(t) - e^{-i|D|t} u(t)\|_{H^s} \leq C\varepsilon^2 \text{ as long as } 0 \leq t \leq \frac{1}{\varepsilon^2 \log \frac{1}{\varepsilon}}\]
Szegő equation

- Hamiltonian equation in $L^2_+(\mathbb{R})$ corresponding to $E(u) = \int |u|^4 dx$
- globally well-posed in $H^s_+$, $s \geq \frac{1}{2}$: $\|u(t)\|_{H^s_+}^{\frac{1}{2}} \leq C$ for all $t$
- Gérard, Grellier (2010): complete integrability - Lax pair:
  \[ \partial_t H_u = [B_u, H_u] , \text{ where } H_u f = \Pi_+(u \bar{f}) \text{ Hankel operator} \]
- conservation laws: $\|H_u^{n-1} u\|_{L^2} \lesssim \|u\|_{L^{2n}} \lesssim \|u\|_{H^s_+}^{\frac{n}{2}}$ for all $n \in \mathbb{N}$
- explicit formula for solutions in term of the spectral data
- Pocovnicu (2011): infinite time growth of high Sobolev norms:
  If $u(0) = \frac{1}{x+i} - \frac{2}{x+2i}$, then
  \[
  u(t, x) = \frac{\alpha_1 e^{i\phi_1(t)}}{x - c_1(t) + i\beta_1} + \frac{1}{t^2} \cdot \frac{\alpha_2 e^{i\phi_2(t)}}{x - c_2 + \frac{i}{t^2}}
  \]
  $\|u(t)\|_{H^s_+} \sim t^{2s-1} \to \infty$ as $t \to \infty$.
- key idea: $H_{u_0}$ has a double eigenvalue
From Szegő equation to NLW

- Gérard, Grellier (2013): Szegő equation on $\mathbb{T}$
  all solutions are quasi-periodic $\implies$ no unbounded orbits

- Growth for Szegő + Approximation $\implies$ relative growth for NLW:
  \[
  \|v(0)\|_{H^s} = \varepsilon, \quad \|v(t)\|_{H^s} \geq \varepsilon \log \frac{1}{\varepsilon} \quad \text{for} \ t = \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}
  \]

- Scaling invariance of NLW ($L^2$-critical) $\implies$ CKSTT-type of growth

- Work in progress (with Gérard, Lenzmann, Raphaël): saturation of the growth of high Sobolev norms $\implies$ information after the growth time

- Open question: growth of high Sobolev norms for the 1-dimensional models of Majda, McLaughlin, and Tabak