Diffusion in high Sobolev spaces for Hamiltonian PDEs

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Nonlinear Schrödinger Equation

- Nonlinear Schrödinger Equation:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u = |u|^2 u, \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \quad u(t,x) \in \mathbb{C} \\ u(0) = u_0 \end{cases}$$

- Limit of the quantum dynamics of many-body systems, model in nonlinear optics, water waves

- Energy and mass conservation:

$$E(u(t)) := \frac{1}{2} \int |\nabla u(t,x)|^2 + \frac{1}{4} |u(t,x)|^4 dx = E(u(0)),$$
$$M(u(t)) := \int |u(t,x)|^2 dx = M(u(0))$$

- If $d = 1$, NLS is completely integrable $\Longrightarrow$ all integer Sobolev norms stay bounded in time

Nonlinear Schrödinger equation on $\mathbb{T}^2$

- Bourgain (1993): If $u(0) \in H^s(\mathbb{T}^2)$ with $s \geq 1 \implies$ there exists a unique global-in-time solution such that $u(t) \in H^s$ for all t
- **Question**: what is the behavior of solutions as $t \to \infty$?
- Bourgain (1996), Staffilani (1997): $\|u(t)\|_{H^s} < Ct^{C(s-1)}$ as $t \to \infty$
- **Further question**: Is there any solution u such that $\sup_t \|u(t)\|_{H^s} = \infty$? What would be the rate of growth?
- **Conjecture** (Bourgain): $\|u(t)\|_{H^s} \ll t^{\varepsilon} \|u(0)\|_{H^s}$ for all $\varepsilon > 0$

Forward energy cascade

- "forward energy cascade": energy moves from lower frequencies to higher and higher frequencies
- growth of high Sobolev norms captures the energy cascade

$$\lim_{t\to\infty} \|u(t)\|_{H^s} = \lim_{t\to\infty} \left\| \langle\xi\rangle^s \mathcal{F}u(t,\xi) \right\|_{L^2} = \infty \text{ for } s \text{ large}$$

- in the physical space: dynamics moves to smaller and smaller scales causing a chaotic behaviour
- growth of high Sobolev norms is the minimal necessary condition for weak turbulence theory
- weak turbulence is the out-of-equilibrium statistics of random waves, it appeared in plasma physics, water waves (Zakharov ('60s))

Partial results

- Bourgain (1995, 1996): infinite time growth for examples of NLS and NLW (specific nonlinearity or specific perturbation of the Laplacian)

- Kuksin (1997): finite time growth for cubic NLS on $\mathbb{T}^d$, $d = 1, 2, 3$ with small dispersion

- CKSTT (2010): Cubic NLS on $\mathbb{T}^2$: For any $s > 1$, $\varepsilon \ll 1$, $K \gg 1$ there exists a solution $u(t)$ and $T > 0$ such that

$$\|u(0)\|_{H^s} \leq \varepsilon \text{ while } \|u(T)\|_{H^s} \geq K$$

- Hani (2011): infinite time growth for NLS on $\mathbb{T}^2$ with a truncated cubic nonlinearity

- Hani, Pausader, Tzvetkov, Visciglia (2013): infinite time growth for cubic NLS on $\mathbb{R} \times \mathbb{T}^d$, $d = 2, 3, 4$

- Guardia, Kaloshin (2012): $\|u(t)\|_{H^s} \geq K \|u_0\|_{H^s}$ for $0 \leq T \leq K^c$

Cubic half wave equation

- Cubic half wave equation:

$$\text{(NLW)} \qquad \begin{cases} i\partial_t v - |D|v = |v|^2 v, & x \in \mathbb{R}, \quad v(t,x) \in \mathbb{C} \\ v(0) = v_0 \end{cases}$$

  where $\mathcal{F}(|D|v)(\xi) = |\xi|\mathcal{F}v(\xi)$

- Majda, McLaughlin, Tabak (1997): one dimensional models of weak turbulence:

$$i\partial_t v - |D|^\alpha v = |D|^{-\beta/4}\Big(\big||D|^{-\beta/4}v\big|^2|D|^{-\beta/4}v\Big), \quad 0 < \alpha < 1$$

- For $v_0 \in H^s(\mathbb{R})$, $s \geq \frac{1}{2}$, NLW has unique global-in-time solution such that $v(t) \in H^s$ for all t

- Pocovnicu (2011): CKSTT-type of result: For any $s > \frac{1}{2}$, $\delta \ll 1$ there exists a solution $v(t)$ of NLW on $\mathbb{R}$ such that

$$\|v(0)\|_{H^s} \leq \delta, \text{ while } \|v(T)\|_{H^s} \geq \frac{1}{\delta} \text{ for } T = \left(\frac{1}{\delta}\right)^{\frac{s}{\alpha}} e^{\frac{2s-1}{s}\left(\frac{1}{\delta}\right)^{\frac{2}{\alpha}}}$$

Resonant dynamics of NLW

- Birkhoff normal form /Renormalization group method yield the resonant dynamics

$$|\xi| - |\xi_1| + |\xi_2| - |\xi_3| = 0$$
$$\xi - \xi_1 + \xi_2 - \xi_3 = 0$$

$\Longrightarrow \xi, \xi_1, \xi_2, \xi_3$ have the same sign

- Resonant dynamics - Szegő equation:

$$i\partial_t u = \Pi_+(|u|^2 u), \text{ where } \mathcal{F}(\Pi_+ f)(\xi) = \mathbf{1}_{\xi \geq 0} \mathcal{F} f(\xi)$$

- Szegő equation was introduced by P. Gérard and S. Grellier in 2008

- Approximation result: NLW and Szegő equation with the same initial condition $v_0 = u_0 \in H^s_+$, of order ε. Then:

$$\|v(t) - e^{-i|D|t} u(t)\|_{H^s} \leq C\varepsilon^2 \text{ as long as } 0 \leq t \leq \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

Szegő equation

- Hamiltonian equation in $L^2_+(\mathbb{R})$ corresponding to $E(u) = \int |u|^4 dx$
- globally well-posed in H^s_+, $s \geq \frac{1}{2}$: $\|u(t)\|_{H^{\frac{1}{2}}_+} \leq C$ for all t
- Gérard, Grellier (2010): complete integrability - Lax pair:

$$\partial_t H_u = [B_u, H_u], \text{ where } H_u f = \Pi_+(u\bar{f}) \text{ Hankel operator}$$

- conservation laws: $\|H_u^{n-1} u\|_{L^2} \lesssim \|u\|^n_{L^{2n}} \lesssim \|u\|^n_{H^{\frac{1}{2}}_+}$ for all $n \in \mathbb{N}$
- explicit formula for solutions in term of the spectral data
- Pocovnicu (2011) : infinite time growth of high Soblev norms:
  If $u(0) = \frac{1}{x+i} - \frac{2}{x+2i}$, then

$$u(t,x) = \frac{\alpha_1 e^{i\phi_1(t)}}{x - c_1(t) + i\beta_1} + \frac{1}{t^2} \cdot \frac{\alpha_2 e^{i\phi_2(t)}}{x - c_2 + \frac{i}{t^2}}$$

$$\|u(t)\|_{H^s_+} \sim t^{2s-1} \to \infty \text{ as } t \to \infty.$$

- key idea: H_{u_0} has a double eigenvalue

From Szegő equation to NLW

- Gérard, Grellier (2013): Szegő equation on $\mathbb{T}$
  all solutions are quasi-periodic $\Longrightarrow$ no unbounded orbits

- Growth for Szegő + Approximation $\Longrightarrow$ relative growth for NLW:

$$\|v(0)\|_{H^s} = \varepsilon, \quad \|v(t)\|_{H^s} \geq \varepsilon \log \frac{1}{\varepsilon} \quad \text{for } t = \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

- Scaling invariance of NLW (L^2-critical) $\Longrightarrow$ CKSTT-type of growth

- Work in progress (with Gérard, Lenzmann, Raphaël): saturation of the growth of high Sobolev norms $\Longrightarrow$ information after the growth time

- Open question: growth of high Sobolev norms for the 1-dimensional models of Majda, McLaughlin, and Tabak