Integral virtual fundamental chains
via finer virtual structures on moduli spaces

Dingyu Yang

Oct 6, 2017
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   \(J : TM \rightarrow TM\) with \(J^2 = -1\) and \(\omega(\ , J )\) Riemannian metric.
   
   (ii) \(u : (S, j, x) \rightarrow (M, J)\) is a \(J\)-curve, if \((du)^{0,1} = 0\).

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3. (i) \(H : M \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) give vector field \(X_H\) by \(dH =: \omega(X_H, \cdot)\).

(ii) Morse theory of \(A_H([x, \hat{x}]) := -\int_D \hat{x}^*\omega - \int_0^1 H(x(t), t) dt\)
gives Floer homology.

(iii) Differential counts perturbed \(J\)-cylinders, or \((J, H)\)-cylinders

\[(du - X_H \otimes dt)^{0,1} = 0.\]
Global invariants via moduli spaces, geom. transversality

Global invariants, GW invariant & Floer homology, are defined:

* Using moduli spaces \( X := \{(\text{perturbed) } J\text{-curves})/\text{symmetry} \).

* Via pulling back the evaluation map \( \text{ev} : X \to Y \), where \( Y \) is a manifold: \( M \times n \), or \( \{\text{closed orbits of } X_H \} \), respectively.

Moduli spaces are not nice spaces in the usual sense:

1. Non-regular: (McDuff:) Simple \( J \)-curves by generic \( J \).
   (Floer-Hofer-Salamon:) (\( J \), \( H \))-cylinders by generic \( t \)-dep \( H \).

2. Noncompact: \( J \)-curves can bubble off. Consider \( \{\text{nodal curves} \} \).

3. (2) \( \implies \) multiple-covers, never by perturbing structures on \( M \).

4. Domain-dep pert makes curve simple, but its compactification leads (3). Ok if (3) only happens in codim-2 (semi-positivity).

5. Divide domain-reparam sym (topology/cptness) \( X \) orbifold-based.

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* Polyfold approach (Hofer-Wysocki-Zehnder, ...)
  (a) From a space $B$ of all candidates, $\{ J\text{-curves}\}/\text{iso}$ is cut out by a section $f$ of Fredholm nature in a bundle $\mathcal{E} \to B$.
  (b) Brand new smoothness, local models (of jumping dimensions), and local implicit function theorem.
  (c) Easily globalize: patch over diffeos, usual genericity argument.

* Kuranishi-type approach (By early 2014, Fukaya-Oh-Ohta-Ono, Joyce, McDuff-Wehrheim, Pardon, Y., ...)
  (a) (Easy local model) $X$ locally cut out by a section $s$ in a f.d. bdle.
  (b) Coordinate change can increase dimension. $\Rightarrow$ No precompact open neighborhood of patched zero sets in patched ambient bases.
  (c) Global topological/algebraic solution for (b), when charts patch.

As of Oct 5, 2017, the output from the above is a virtual fund.

chain/cycle over $\mathbb{Q}$, which is a regular replacement of bad compact zero set $f^{-1}(0)$ or $\bigcup s^{-1}(0)$.

$\bigcup$ is the patching identification.

Because the symmetry and transversality don't get along $\Rightarrow$ Need symmetric $\mathbb{Q}$-weighted branchwise multi-sections, or equiv.
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As of Oct 5, 2017, the output from the above is a virtual fund. chain/cycle over $\mathbb{Q}$, which is a regular replacement of bad compact zero set $f^{-1}(0)$ or $\bigsqcup_i s_i^{-1}(0)/\gamma$. $\gamma$ is the patching identification. Because the symmetry and transversality don’t get along $\implies$ Need symmetric $\mathbb{Q}$-weighted branchwise $\amalg$ multi-sections, or equiv.
Integral virtual fundamental chains (joint w/ Guangbo Xu)

Example: 2-sphere $S$ w/ an orbifold point $\{z\} := S^{\mathbb{Z}_3}$ of symm. $\mathbb{Z}_3$.

1. It fits as a special case of the previous polyfold/Kuranishi theories, and it is already regular and no room in the 0-bundle to do much else (like perturbation). Euler char. rational.

2. But we know it has no hole, whose information can be captured, and indeed, $S\setminus\{z\}$ is a pseudocycle, and we count 2 for this “virtual” (pseudo-)cycle.

3. The subspace of points with nontrivial stabilizers being of codimension 2 (in the base) is not enough: We still need to regularize, and face incompatibility of symm. & transversality.

4. Another aspect is that the normal bundle $N_{S^{\mathbb{Z}_3}}S$ has a complex structure in this example, which generalizes.
A good coordinate system GCS (of 2 charts)

Any Kuranishi-type theory, you can get a finitely many charts covering moduli $X$, a good coordinate system. Consider 2 charts (4 generalizes). $C_I := (s_I : U_I \to E_I, \psi_I : s_I^{-1}(0)/\Gamma_I \to X; \Gamma_I}$ and one for $J$. 
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A coordinate change $C_I \to C_J$ says:

(2) (Enough) open part $C_I|_{U_{JI}}$ of $C_I$ embeds in $C_J$, intertwining all the data. So $s_I|_{U_{JI}}$ sits in as part of $s_J$. Images denoted by $\tilde{\cdot}$.

(2) $U_J$ can have larger dimension. The extra direction is ‘cancelled out’ by $ds_J$ matching normals of $\tilde{U}_{JI}$ in $U_J$ with fiber normals of $\tilde{E}_I$ in $E_J$”, a canonical condition at zeros $s_J|_{\tilde{U}_{JI}}$. 
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One way to go up dimension is by improving (2) into condition over $W_{JI}$, a tubular neighborhood projecting onto $\tilde{U}_{JI}$ (remembering the fiber linear structure, containing no extra zeros), and extend $\tilde{E}_J$ to $\tilde{E}_{JI}$ over $W_{JI}$, and asking $s_J|_{W_{JI}}/\tilde{E}_{JI} \pitchfork 0$. A $\pitchfork$ perturbation of $s_I$ is immediately lifted to a $\pitchfork$ perturbation over $W_{JI}$. After things are in the same setting in $E_J$, one can use relative tranversality.
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Theorem (Y., Feb 2014)

*Can do it globally on GCS via \( \exists \) (! up to refinem’t) of level-1 str.*
Between two group-fixed parts in a chart

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Integral virtual fundamental chains/pseudocycles

Need notion of compatible structure involving $W_{jL}^H \to U_{jL}^H$ called group-normal str. providing a common setting at each inductive layer, & a canonical notion of group-normal complex cond./str. $\mathcal{J}$. 

Theorem (Guangbo Xu - Y.)

* Given a group-normal complex $\mathcal{J}$, a $\mathcal{J}$-compatible group-normal structure exists, up to cobordism/refinement.

* ∃(! up to cob) a single-valued equivariant fiberwise polynomial perturbation \( \{\tilde{s}_I\} \) with zero set \( \tilde{X} := \bigsqcup_I \tilde{s}_I - 1(0) / \bigcup \), s.t. space \( \hat{\tilde{X}}[L] \) of points with exact $[L]$-stabilizers is a manifold (with compactification \( \hat{\tilde{X}}[L] \)), \( \tilde{X}[L] \setminus \hat{\tilde{X}}[L] \) has codim $R \geq 2$ (even) and is covered by maps from manifolds fibering over \( \hat{\tilde{X}}[H] \) for all $L \leq H$ (up to conj. & ident.).

* "Floer chain"- and GW-moduli spaces for general symplectic manifolds have group-normal complex str. $\mathcal{J}$, up to quasi-iso/cob.

* Using the stabilizer-free moduli spaces $\hat{\tilde{X}}[\{Id\}]$ with $L = \{Id\}$, the $\mathbb{Z}$-VFC, Floer homology and GW are well-defined over $\mathbb{Z}$. 

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Some further directions

Other part worth exploring:

1. How to make use of $\tilde{M}^{[L]}$ for other groups $[L]$ with $L \neq \{Id\}$. Maybe use pseudocycle stratification to relate to homotopy quotient defined using universal family-dependent $J$ in certain situations.

2. Since multiple branch-covers can be stabilizer-free, so integral GW (defined for all, not just for CY3) is not GV. What is the geometric meaning on the curve counting side?

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6. Find alternative condition (to group-normal complex) applicable to other moduli spaces of geometric PDE.
Thank you!