Analysis and topology on arithmetic locally symmetric spaces

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Basic example

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For now:

- Γ is this group or a finite index congruence subgroup, and $M = \mathbb{H}/\Gamma$, an "arithmetic locally symmetric space."
- M' is a small perturbation of M, e.g. $\mathbb{H}/\Gamma'$ for a generic Γ' (nothing to do with integers).

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- Then we will talk about some curious topological features, which actually are rather parallel to the analytic features above.
- To conclude, I will discuss how the topology of these spaces is related to algebraic geometry, and describe some of the issues which I hope to study over the course of this year.

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- On $\mathbb{H}$ the Riemannian Laplacian is given by $-y^2(\partial_{xx} + \partial_{yy})$.
- On $L^2(\mathbb{H}/\Gamma)$ this has infinitely many eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and they satisfy Weyl's law : their mean spacing is $\frac{4\pi}{\text{area}}$.

Some eigenvalues

Here are 27 eigenvalues after $640,000$, as computed by H. Then:

$$1.1, 8.8, 56.3, 76.5, 77.4, 107.8, 111.6, 120.6, 121.3,$$

$$132.0, 134.3, 134.8, 154.4, 156.15, 158.8, 166.6, 202.4, 207.4, 216.0$$

$$218.07, 225.02, 231.28, 266.36, 272.17, 296.53, 310.28, 316.29$$

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The mean spacing is $12 = \frac{4\pi}{\text{area}}$ according to Weyl's law. Here is a picture; do you notice anything surprising?

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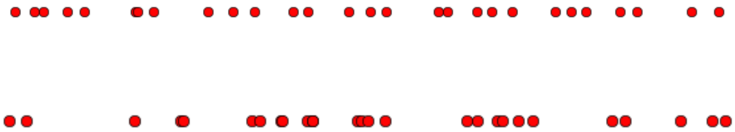

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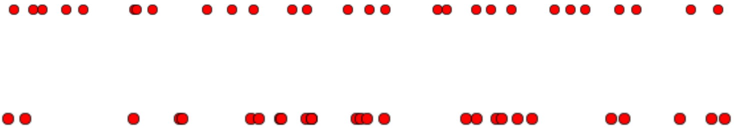

- Eigenvalues repel! Two in an interval of length ε with probability $\sim \varepsilon^3$; k of them with probability $\sim \varepsilon^{k(k+1)/2}$.

In fact, it is surprising that there exist eigenvalues *at all*, because $\Gamma\backslash\mathbb{H}$ is noncompact.

- To show the existence of eigenvalues for the modular surface, Selberg introduced the trace formula. His proof applies only to Γ used special properties of the Riemann ζ-function;

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- To show the existence of eigenvalues for the modular surface, Selberg introduced the trace formula. His proof applies only to Γ used special properties of the Riemann ζ-function;
- After the work of Phillips and Sarnak it is generally believed that a small deformation Γ' of Γ *destroys all eigenvalues*, i.e. there are no Laplacian eigenfunctions at all in $L^2(\mathbb{H}/\Gamma')$.

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- The surface M has a certain class of hidden symmetries, the "Hecke operators."
- These reduce the influence of one eigenvalue on another.

What is a Hecke operator

- The map $z \mapsto pz$ doesn't give a map $M \to M$, but it almost does:
- For each prime p we have a multi-valued function $T_p : M \to M$:

$$T_p(z) = \{z_1, \ldots, z_{p+1}\}.$$

Locally, each map $z \mapsto z_i$ is isometric.

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- More generally, if Γ is an arithmetic subgroup of a semisimple Lie group – e.g. $\mathrm{SL}_n(\mathbf{Z}), \mathrm{Sp}_{2g}(\mathbf{Z})$ – then Γ acts on a canonical space of curvature ≤ 0, the Riemannian symmetric space $\mathcal{H}$.

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- An arithmetic locally symmetric space is any such quotient $\mathcal{H}/\Gamma$. It has a canonical Riemannian structure. Many natural spaces arise thus.

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Lineare Substitutionen mit ganzen complexen Coefficienten II. 361

e) $\left(\xi - \frac{1}{2}\right)^2 + \left(\eta - \frac{\sqrt{D}}{2}\right)^2 + \zeta^2 = \frac{1}{2^2}$,

Tipo I) $a_1 = 1, \quad a_2 = 1, \quad c_1 = 2, \quad b_1 = -\frac{D}{2}$,

f) $\xi^2 + \left(\eta - \frac{D-1}{2\sqrt{D}}\right)^2 + \zeta^2 = \frac{1}{2^2 D}$,

Tipo II) $a_2 = 0, \quad a_1 = 1 - D, \quad c_1 = 2, \quad b_1 = 1 - \frac{D}{2}$.

Le sfere di riflessione qui indicate a), b), c), d), e), f) bastano già per i piccoli valori di D a separare il poliedro P cercato,

§ 12.

Il gruppo $\overline{\Gamma}^{(i)}$.

Benchè i casi $D = 1$, $D = 3$ siano già stati trattati nel lavoro precedente, non sembra qui inutile coordinare la determinazione dei poliedri fondamentali corrispondenti alle osservazioni generali del paragrafo precedente.

Se $D = 1$, si considerino i tre piani di riflessione

(1) $\xi = \frac{1}{2}$, (2) $\eta = 0$, (3) $\xi - \eta = 0$

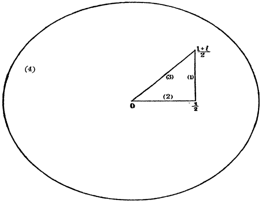

Fig.a 1a

e si indichi con P il poliedro racchiuso in R da questi tre piani esternamente alla sfera

(4) $$\xi^2 + \eta^2 + \zeta^2 = 1.^{*)}$$

*) In questa come nelle figure seguenti si osservano le traccie sul piano $\xi\eta$ dei piani e delle sfere di riflessioni numerati come nel testo.

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- We examine the simplest topological invariant:

$$H_1(M, \mathbf{Z}) \simeq \Gamma^{\mathrm{ab}}.$$

Some early computations by Elströdt, Mennicke, Grunewald and Grunewald, Schwermer for subgroups $\Gamma_0(n)$ of the Bianchi group It was (relatively) recently that we can easily compute enough examples to see something interesting.

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- $\Gamma_0(118+175i)^{\mathrm{ab}} = \mathbf{Z} \oplus T$ where $|T| > 10^{310}$.

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$$\frac{\log\left(\# H_1(M, \mathbf{Z})_{\mathrm{tors}}\right)}{\mathrm{vol}(M)} \to \frac{1}{6\pi}.$$

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Anyway, let us look at some data computed by Brock -Dunfield on how this conjecture shapes up for arithmetic versus nonarithmetic M.

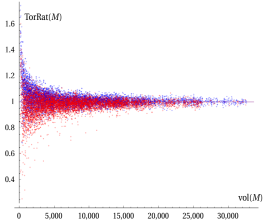

Figure 4.4. Congruence covers of arithmetic twist-knot orbifolds. The blue dots are covers where $b_1 = 0$ and the red dots covers where $b_1 > 0$.

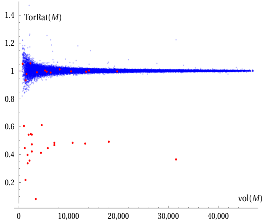

Figure 4.5. Congruence covers of nonarithmetic twist-knot orbifolds; as before, blue dots indicate $b_1 = 0$ and red dots $b_1 > 0$.

Repulsion of mod p classes

In topology there is a surprising parallel to "repulsion of eigenvalues."

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- By contrast – eyeballing data – factors of $(\mathbf{Z}/p\mathbf{Z})^k$ with $k \gg 1$ are much more frequent for *arithmetic* M. Again, this should be attributed to the influence of Hecke operators.

Summary

In both the analytic and topological case, the distribution of eigenvalues/homology is controlled by a certain linear map: the Laplacian, or the differential in the chain complex. These can be modeled by random symmetric or p-adic matrices in general; but being forced to commute with Hecke operators causes rigid and unusual behavior.

Return to $M = \mathbb{H}/\Gamma$.

- This M has the structure of an algebraic curve over $\mathbb{Q}$, i.e. $M = \mathbf{X}(\mathbf{C})$ for $\mathbf{X} \subset \mathbb{P}^N_{\mathbb{Q}}$.

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- or more succinctly

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- The correct context to take these virtual combinations is the theory of pure motives:

$$\text{algebraic varieties} \hookrightarrow \text{pure motives}$$

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- The traces of T_p on H^1 and H^2 are the same. So the right hand side cannot be related to a Lefschetz number.

Some things I'm thinking about this year

While Shimura varieties (e.g. $\mathbb{H}/\Gamma$) are far better understood, the general arithmetic locally symmetric spaces (e.g. $\mathbb{H}^3/\Gamma$) actually exhibit richer structures in their topology, e.g.:

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(d) The relationship of M not just to pure motives but to *mixed* motives.

In fact there is one case of (d) that has been around for a long time: the algebraic K-theory of $\mathbb{Z}$, reflecting mixed Tate motives, is related to the *stable* homology of the $\mathrm{SL}_n(\mathbb{Z})$ symmetric space.