Rota's conjecture and algebraic cycles in permutohedral varieties

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A *matroid* on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors:

1. Every subset of an independent set is an independent set.
2. If an independent set A has more elements than independent set B, then there is an element in A which, when added to B, gives a larger independent set.

- Let V be a vector space over a field k, and E a finite set of vectors.

  Call a subset of E independent if it is linearly independent.

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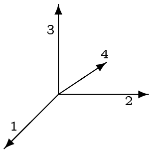
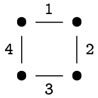
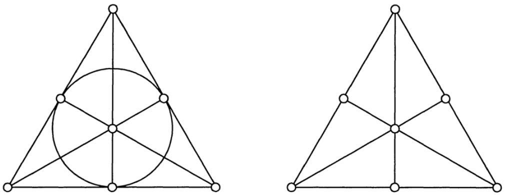
- Let G be a finite graph, and E the set of edges.

  Call a subset of E independent if it does not contain a circuit.

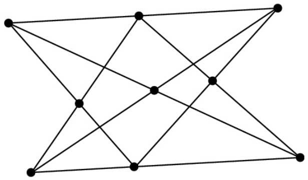
  This defines a *graphic matroid* M.







Fano matroid is realizable iff char$(k) = 2$. Non-Fano matroid is realizable iff char$(k) \neq 2$.



Non-Pappus matroid is realizable over no field.

- The *chromatic polynomial* of G is the function

$$\chi_G(q) = (\text{number of proper colorings of } G \text{ with } q \text{ colors}).$$

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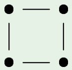
Example



$$\chi_G(q) \quad = \quad 1q^4 - 4q^3 + 6q^2 - 3q$$

The same polynomial is defined for any matroid.

Conjecture (Rota, Welsh, Read, Mason. . .)

The coefficients of the chromatic polynomial $\chi_M(q)$ form a log-concave sequence for any matroid M:

$$a_{i-1}\, a_{i+1} \le a_i^2 \quad \textit{for all } i.$$

Let X be an algebraic variety over k.

Definition

A homology class ξ of X with real coefficients is said to be *prime*

if some positive multiple of ξ is the class of a subvariety.

Define

$$\mathscr{P}_d(X) := \Big(\text{the closure of the set of } d\text{-dimensional prime classes}\Big).$$

Example

If $X = \mathbb{P}^m \times \mathbb{P}^n$, then

$$H_{2d}(X;\mathbb{R}) = \Big\{ \xi = \sum_i x_i[\mathbb{P}^{d-i} \times \mathbb{P}^i] \mid x_i \in \mathbb{R} \Big\}.$$

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Then $\mathscr{P}_d(X)$ is the set

$$\Big(\{x_i\} \text{ is a } \textit{log-concave} \text{ sequence of } \textit{nonnegative} \text{ integers with } \textit{no internal zeros} \Big).$$

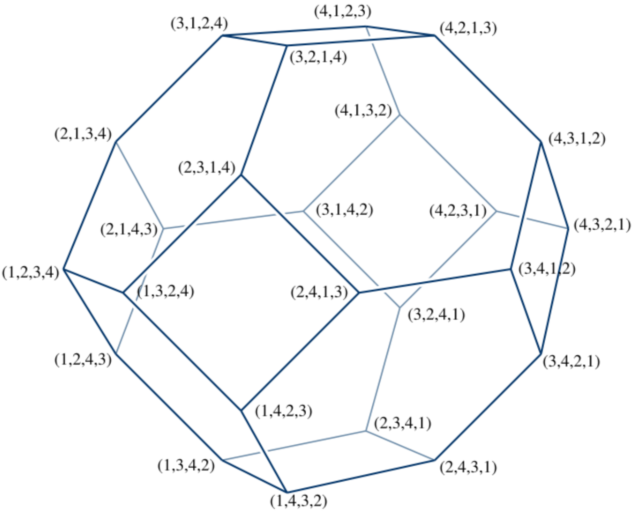
This structure is *not* visible if we work with integral homology classes.

For example, there is no subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ with the homology class

$$1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5],$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros.

Matroid theory is a study of the toric variety X_{A_n} of the n-dimensional *permutohedron*:



Corresponding to a matroid M of size n and rank r,

there is an r-dimensional integral homology class of X_{A_n}, denoted Δ_M.

The homology class Δ_M determines M, and we know exactly which homology classes in X_{A_n} are of this form.

Theorem

Let k be a field, and M be a matroid of size n.

(i) *The homology class Δ_M is effective in X_{A_n} over k.*

(ii) *The homology class Δ_M is the class of a subvariety in X_{A_n} over k*

if and only if M is realizable over k.

Theorem

Under the "anticanonical" map

$$\pi : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

the matroid homology class pushforwards

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Since prime classes map to prime classes, the log-concavity conjecture is true

for all matroids which are realizable over some field.

In our language, the log-concavity conjecture is equivalent to

Conjecture

For any matroid M and any field k,

$$\pi_*(\Delta_M) \in \mathscr{P}_r(\mathbb{P}^n \times \mathbb{P}^n).$$

If the conjecture is true, it might be because the same is true in X_{A_n}.

Speculation

For any matroid M and any field k,

$$\Delta_M \in \mathscr{P}_r(X_{A_n}).$$

In other words, all matroids are realizable over every field (in a generalized sense).